Filtrations and Induced Maps

Lecture 8 - CMSE 890

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::
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Goals

Goals for today:

- Filtrations
- Betti curves
- Induced Maps

Section 1

Filtrations

Simplicial Complex Filtration

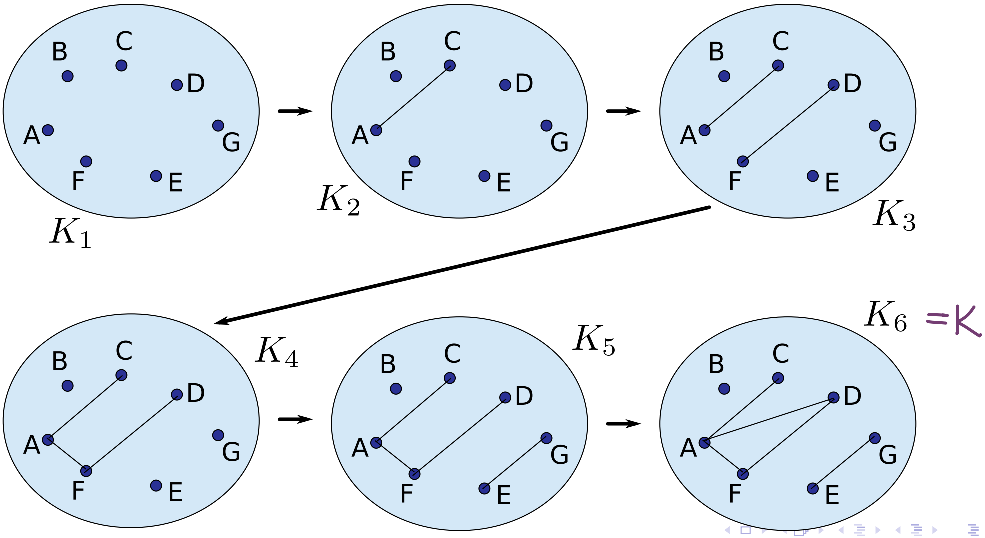
Definition

A filtration $\mathcal{F} = \mathcal{F}(K)$ of a simplicial complex K is a nested sequence of its subcomplexes

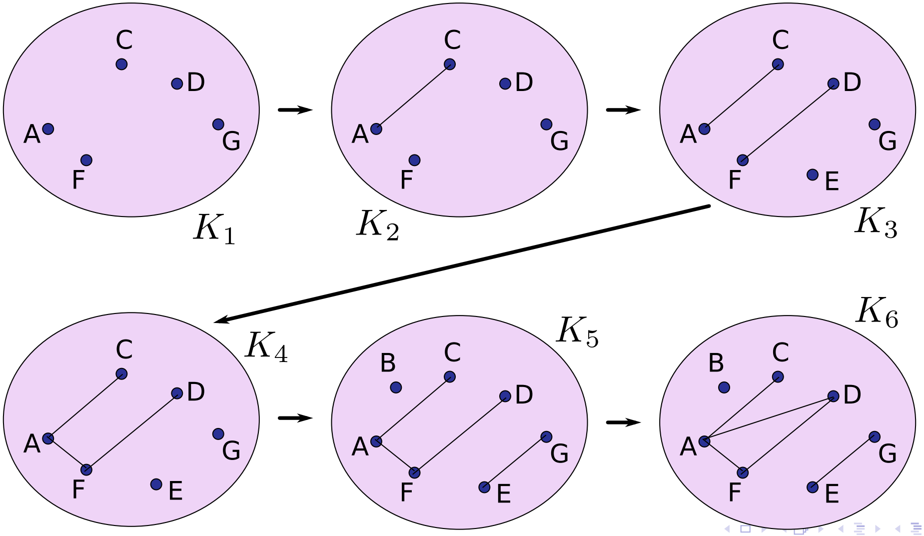
$$\mathcal{F} : \emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K.$$

$\mathcal{F}$ is called *simplex-wise* if $K_i \setminus K_{i-1}$ is empty or a single simplex.

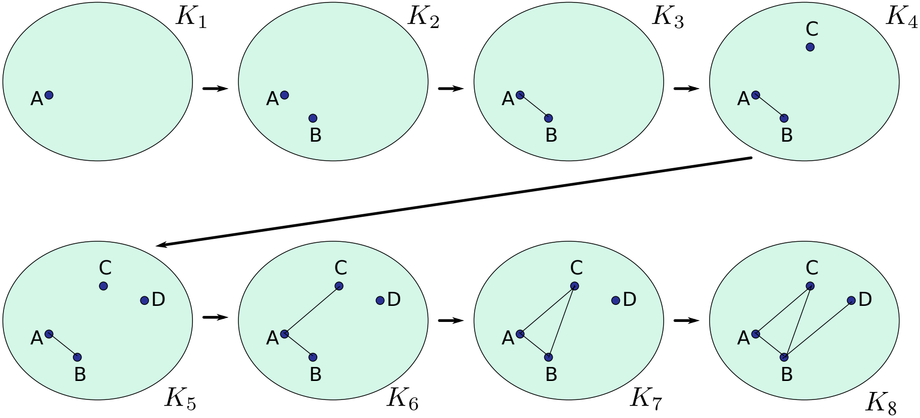
Example 1



Example 3



Example 3



~~Simplex-wise~~ monotone function

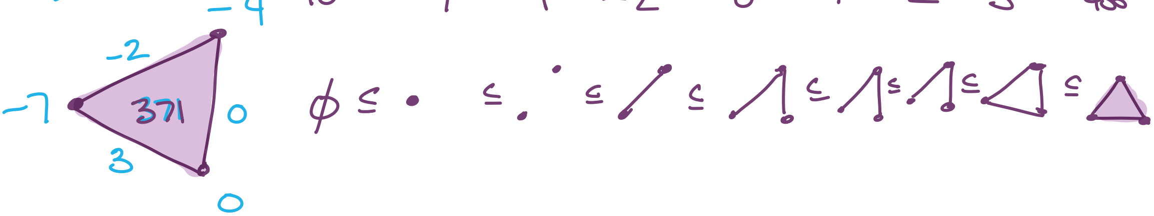
→ so that K_i always a simp. cpx.

A simplex-wise function $f : K \to \mathbb{R}$ is called monotone if for every $\tau \leq \sigma$, $f(\tau) \leq f(\sigma)$. Fix values $a_0 \leq a_1 \leq \cdots a_n$. Let $K_i = f^{-1}(-\infty, a_i]$. The sublevelset filtration induced by this function is defined to be

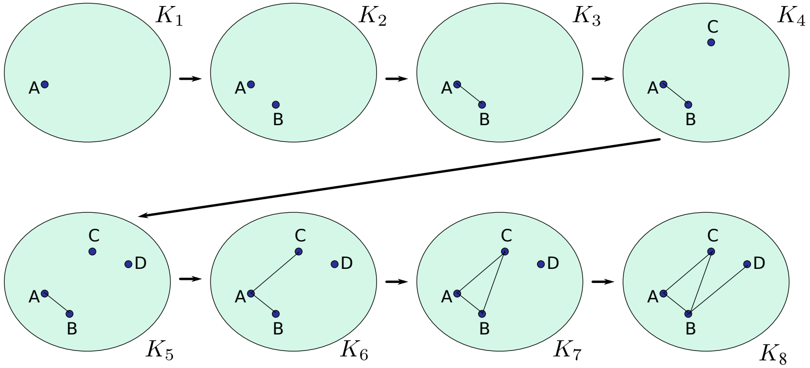
$$\emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K.$$

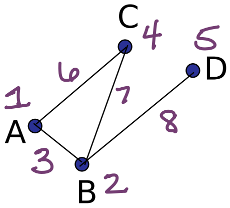
$f : K \to \mathbb{R}$

a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
-10	-7	-4	-2	0	1	2	3	400

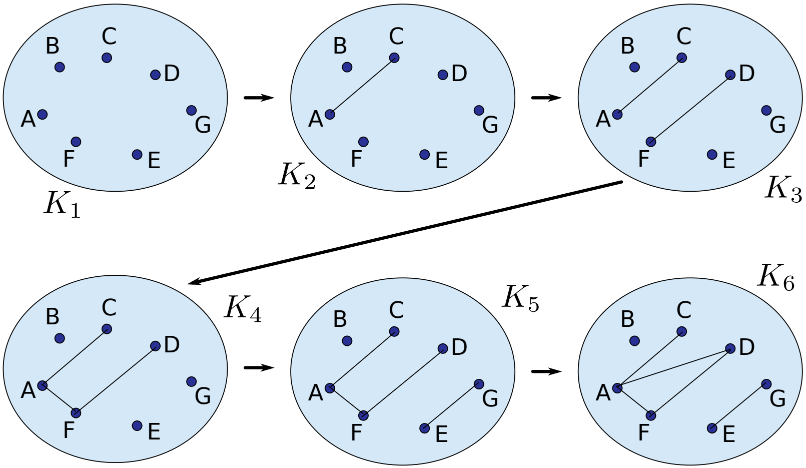


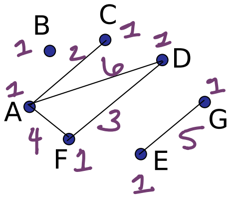
Example: Function inducing filtration





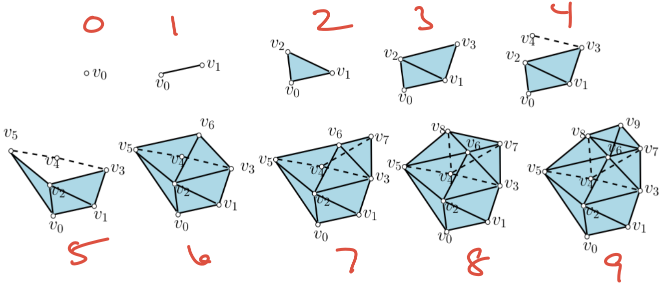
Tryit: Function inducing filtration





Everything comes from a function

Vertex valued function (lower star filtration)



$St(v) = \{\sigma \in K \mid v \in \sigma\}$

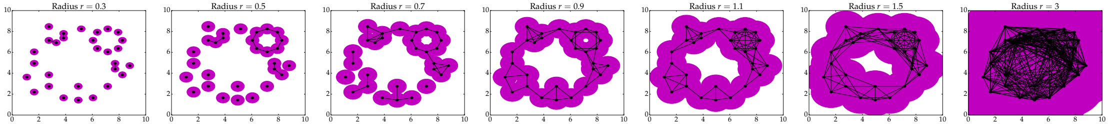
$St\_(v) = \{\sigma \in K \mid v \in \sigma, f(\sigma) \leq f(v)\}$

- Given $f : V \to \mathbb{R}$ defined on vertices of K.
- Extend to simplices by $f(\sigma) = \max_{v \in \sigma} f(v)$.

Everything comes from a function

Rips complex filtration

$VR(r) = \{\sigma \subset P \mid \|v-w\| \leq 2r \quad \forall v,w \in \sigma\}$



$K =$ complete simplicial complex on n-vertices

$VR(0) = V$

$$f: K \longrightarrow \mathbb{R}$$

$$v \longmapsto 0 \quad \forall v \in V$$

$$e \longmapsto \|v-w\| \qquad e = vw$$

$$\sigma \longmapsto \max_{v,w \in \sigma} \|v-w\|$$

Section 2

What to do with a filtration: Betti Curves

not Cont !

not curve!

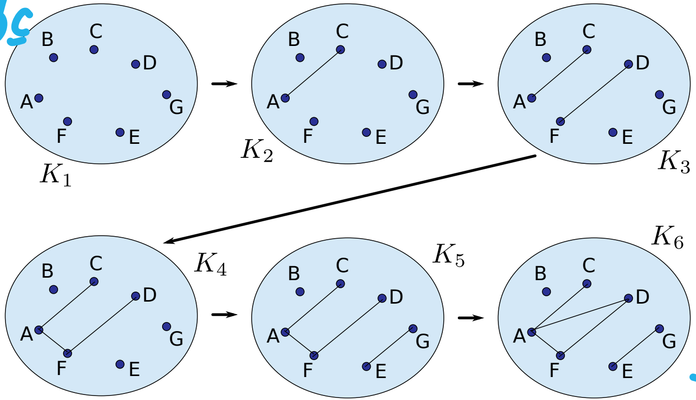
Betti curve

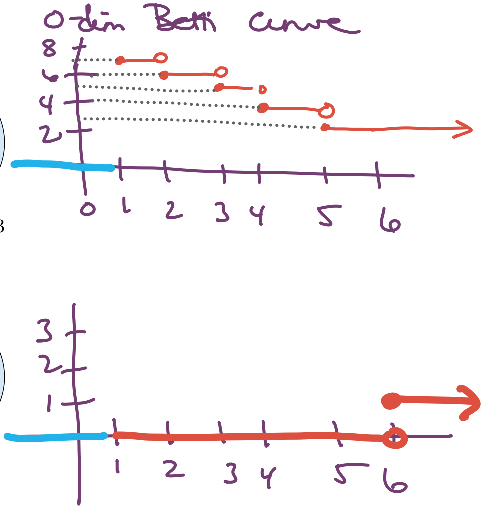
The p-th Betti curve is a function

$$\begin{aligned} \beta_p(\mathcal{F}) : \overset{\mathbb{R}}{\cancel{\mathbb{Z}}} &\to \mathbb{Z} \\ \overset{a_i}{\cancel{i}} &\mapsto \beta_p(K_i) = \mathrm{rk}(H_p(K_i)) \end{aligned}$$

Example: Betti curves

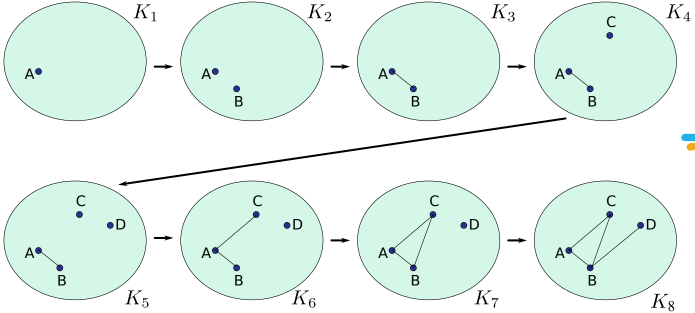
Write the 0th and 1st Betti curves for this filtration

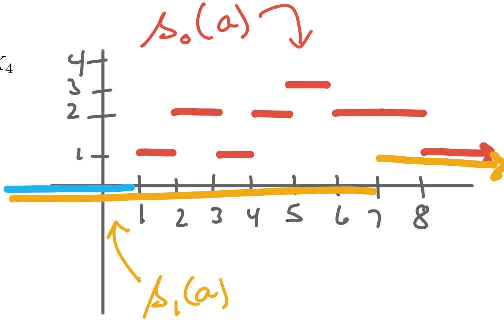




Tryit: Betti curves

Write the 0th and 1st Betti curves for this filtration



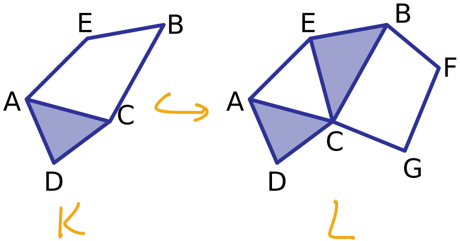


Section 3

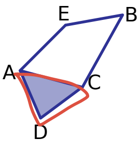
Induced maps

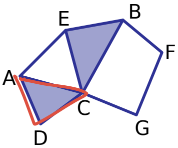
Simplicial maps

A simplicial map between abstract simplicial complexes $f : K \to L$ is induced by a map on the vertices $f : V(K) \to V(L)$.
(In this class, we care about inclusions.....)



Big diagrams of vector spaces, $f_\#$





$AC + CD + AD$

α

$$
\begin{tikzcd}
\cdots \arrow[r] & C_2(K) \arrow[r, "\partial_2^K"] \arrow[d, "f_\#"] & C_1(K) \arrow[r, "\partial_1^K"] \arrow[d, "f_\#"] & C_0(K) \arrow[r, "\partial_0^K"] \arrow[d, "f_\#"] & 0 \\
\cdots \arrow[r] & C_2(L) \arrow[r, "\partial_2^L"] & C_1(L) \arrow[r, "\partial_1^L"] & C_0(L) \arrow[r, "\partial_0^K"] & 0
\end{tikzcd}
$$

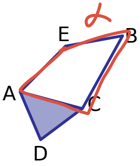
$(f(\alpha))$

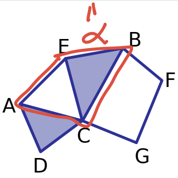
α

$AC + CD + AD$

Induced map

$f_\#(\alpha)$



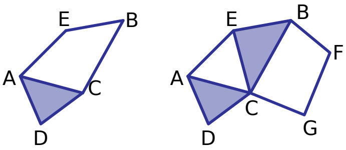


Given a simplicial map $f : K \to L$, the induced map on homology is defined by

$$\begin{array}{cccc} f_* : & H_p(K) & \to & H_p(L) \\ & [\alpha] & \mapsto & [f_\#(\alpha)] \end{array}$$

$\alpha = AE + EB + CB + AC$

Commutative diagram



A diagram commutes if any path of functions are equal.

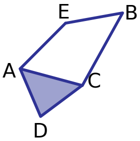
AC+CD $\longmapsto$ A+D

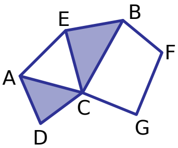
$$\begin{tikzcd}
\cdots \arrow[r] & C_2(K) \arrow[r, "\partial_2^K"] \arrow[d, "f_\#"] & C_1(K) \arrow[r, "\partial_1^K"] \arrow[d, "f_\#"] & C_0(K) \arrow[r, "\partial_0^K"] \arrow[d, "f_\#"] & 0 \\
\cdots \arrow[r] & C_2(L) \arrow[r, "\partial_2^L"] & C_1(L) \arrow[r, "\partial_1^L"] & C_0(L) \arrow[r, "\partial_0^K"] & 0
\end{tikzcd}$$

AC+CD $\longmapsto$ A+D

Claim: $f_\# \circ \partial^K = \partial^L \circ f_\#$.

Effect on cycles





Cycles go to cycles!

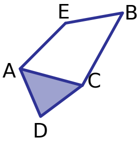
$AC+CD+AD \mapsto 0$

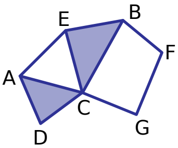
$$\begin{tikzcd}
\cdots \arrow[r] & C_2(K) \arrow[r, "\partial_2^K"] \arrow[d, "f_\#"] & C_1(K) \arrow[r, "\partial_1^K"] \arrow[d, "f_\#"] & C_0(K) \arrow[r, "\partial_0^K"] \arrow[d, "f_\#"] & 0 \\
\cdots \arrow[r] & C_2(L) \arrow[r, "\partial_2^L"] & C_1(L) \arrow[r, "\partial_1^L"] & C_0(L) \arrow[r, "\partial_0^K"] & 0
\end{tikzcd}$$

$AC+CD+AD \mapsto 0$

$0 = 0$

Effect on boundaries





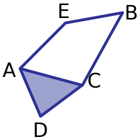
Boundaries go to boundaries!

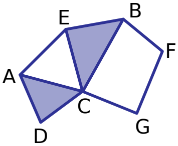
$ACD \longmapsto AC + CD + AD$

$$\begin{tikzcd}
\cdots \arrow[r] & C_2(K) \arrow[r, "\partial_2^K"] \arrow[d, "f_\#"] & C_1(K) \arrow[r, "\partial_1^K"] \arrow[d, "f_\#"] & C_0(K) \arrow[r, "\partial_0^K"] \arrow[d, "f_\#"] & 0 \\
\cdots \arrow[r] & C_2(L) \arrow[r, "\partial_2^L"] & C_1(L) \arrow[r, "\partial_1^L"] & C_0(L) \arrow[r, "\partial_0^K"] & 0
\end{tikzcd}$$

$ACD \longmapsto AC + CD + AD$

Repeat: Induced map

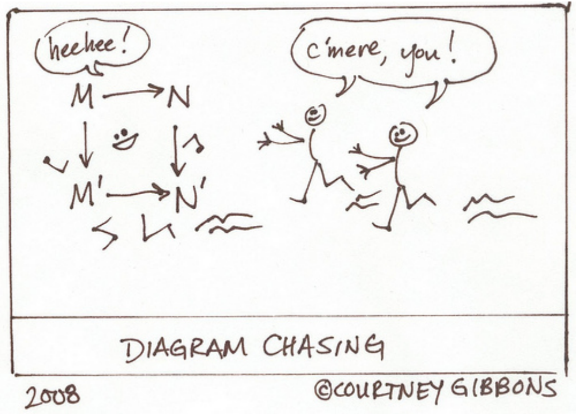




Given a simplicial map $f : K \to L$, the induced map on homology is defined by

$$\begin{array}{rccc} f_* : & H_p(K) & \to & H_p(L) \\ & [\alpha] & \mapsto & [f_\#(\alpha)] \end{array}$$

Diagram Chasing



Try it:

$[\alpha_1] \in H_1(L)$

$[\alpha_3] = [\partial(EFL + LFG)]$

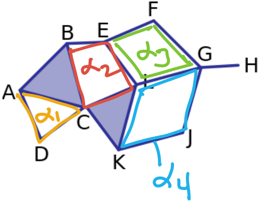
If $H_1(K)$ generated by

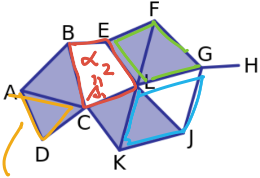
$$[\alpha_1] = [AC + CD + AD]$$
$$[\alpha_2] = [BE + EL + CL + CB]$$
$$[\alpha_3] = [EF + FG + GL + EL]$$
$$[\alpha_4] = [GL + GJ + JK + LK]$$

and $H_1(L)$ generated by

$$[\beta_1] = [BE + EL + CL + CB]$$
$$[\beta_2] = [GL + GJ + LJ]$$

write the matrix representing $f_* : H_1(K) \to H_1(L)$ induced by inclusion.

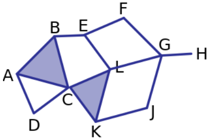


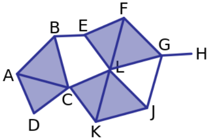


$[\alpha_1] = [0]$

$$\begin{array}{c|cccc} & [\alpha_1] & [\alpha_2] & [\alpha_3] & [\alpha_4] \\ \hline [\beta_1] & 0 & 1 & 0 & 0 \\ [\beta_2] & 0 & 0 & 0 & 1 \end{array}$$

More spare paper





$$\begin{tikzcd}
\cdots \arrow[r] & C_2(K) \arrow[r, "\partial_2^K"] \arrow[d, "f_\#"] & C_1(K) \arrow[r, "\partial_1^K"] \arrow[d, "f_\#"] & C_0(K) \arrow[r, "\partial_0^K"] \arrow[d, "f_\#"] & 0 \\
\cdots \arrow[r] & C_2(L) \arrow[r, "\partial_2^L"] & C_1(L) \arrow[r, "\partial_1^L"] & C_0(L) \arrow[r, "\partial_0^K"] & 0
\end{tikzcd}$$

Next time: Persistence module

Given a filtration $\mathcal{F}$, the p-dimensional persistence module is

$$H_p(\mathcal{F}) : 0 = H_p(K_0) \to H_p(K_1) \to H_p(K_2) \cdots \to H_p(K_n) = H_p(K)$$

with maps induced by inclusion.

For next time

- No class 9/23 or 9/25
- Everyone homework:
    - Find the AATRN youtube channel
    - Find one of the Tutorial-a-thon playlists
    - Watch one video (or more, they're largely less than 15 minutes long). Write a few sentences about what you learned and any questions you have and post it with a link to the video on our slack channel.
    - Comment on at least one other person's post.