Simplicial Complexes

Lecture 3 - CMSE 890

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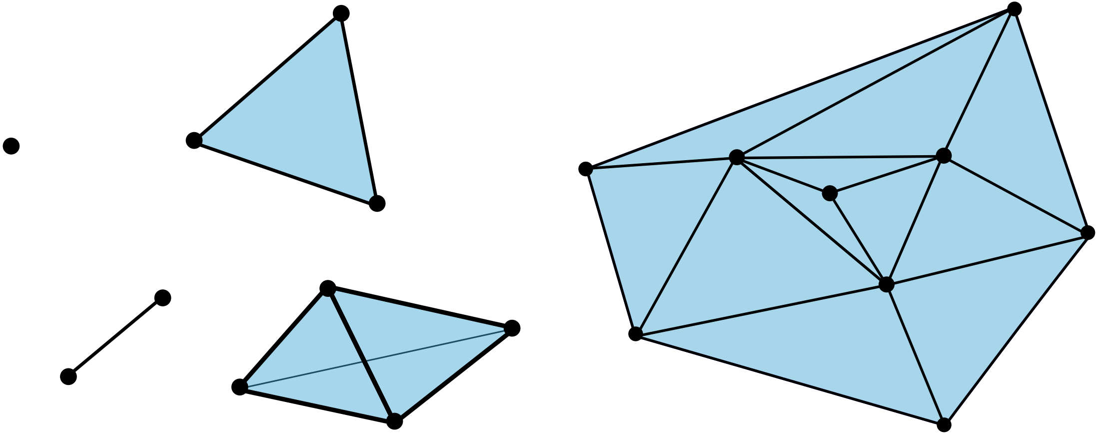
Goals

Goals for today: Ch 2.1

- Define geometric/abstract simplicial complexes
- Define simplicial maps

Idea

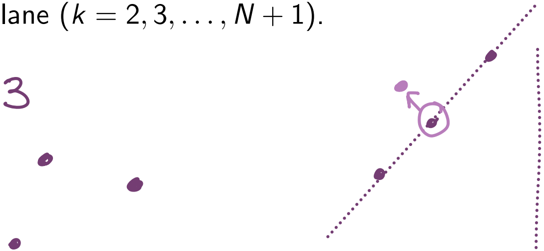
A simplicial complex is a generalizion of a graph



General position

A set $\{a_0, \cdots, a_n\} \subset \mathbb{R}^N$ is **in general position** if no k of them lie in a $k-2$ dimensional hyperplane ($k = 2, 3, \ldots, N+1$).

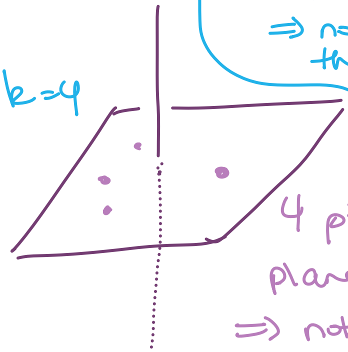
$k=3$



Not on the same line (1-dim hyperplane) they are in gen pos.

$k=2$, 0-dim hyper $\Rightarrow$ not at the same point

$k=4$



4 pts on plane $\Rightarrow$ not in gen. pos

Warning: Papers will often modify this definition to suit their purposes!

Given a set of points $\{a_0, \cdots, a_n\}$ in general position, the n-plane P spanned by the points is

$\subseteq \mathbb{R}^N$

$$P = \left\{ \underbrace{\sum_{i=0}^{n} t_i a_i}_{t_0 a_0 + t_1 a_1} \in \mathbb{R}^N \mid \underbrace{\sum t_i}_{t_0 + t_1} = 1 \right\}.$$

$\boxed{n=1}$

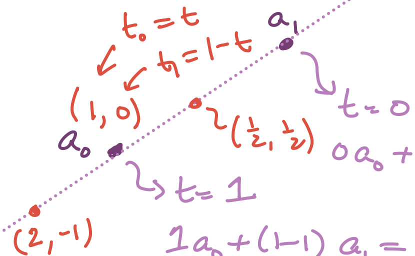
$\{a_0, a_1\}$

$t_0 a_0 + t_1 a_1$

$t\, a_0 + (1-t)\, a_1$

$t_0 + t_1 = 1$

$t + (1-t) = 1$



$0 a_0 + (1-0) a_1 = a_1$

$1 a_0 + (1-1)\, a_1 = a_0$

$n = 2$

$t_0 + t_1 + t_2 = 1$

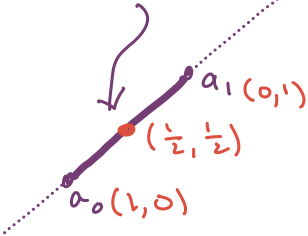
n-simplex

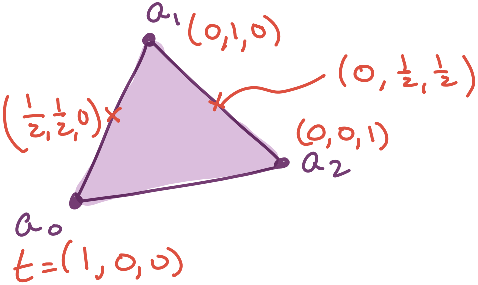
Given a set of points $\{a_0, \cdots, a_n\}$ in general position, the n-simplex σ spanned by the points is

$$P = \left\{ \sum_{i=0}^{n} t_i a_i \in \mathbb{R}^N \mid \sum t_i = 1 \right\}$$

such that $t_i \geq 0\ \forall i$.

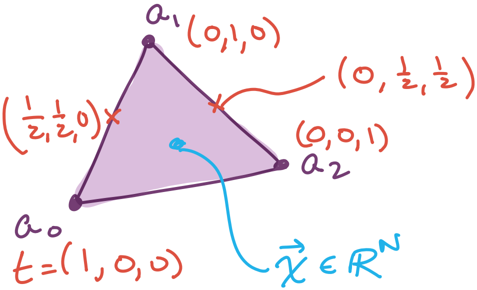
$t_0 a_0 + t_1 a_1 + t_2 a_2$





Barycentric coordinates

The numbers t_i are uniquely determined ~~by x~~ and are called **barycentric coordinates**.



- $\{a_0, \cdots, a_n\}$ are called the **vertices** of σ. n-simplex
- The dimension of $\sigma = [a_0, \cdots, a_n]$ is n.
- Any simplex spanned by a subset of $\{a_0, \cdots, a_n\}$ is a **face** of σ.
- A face is proper if it isn't the simplex itself.

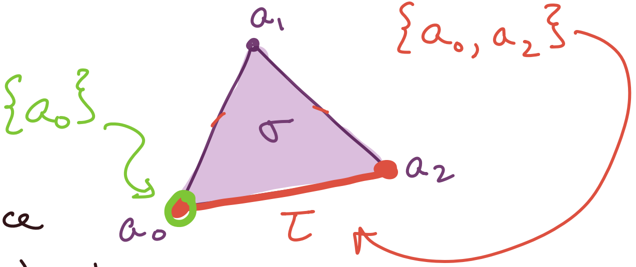
Notation

$\tau \leq \sigma$

"is a face of"

$\sigma \leq \sigma$ not proper

$\tau < \sigma$

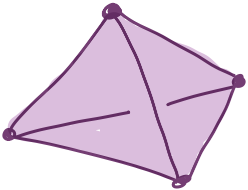


- A codimension-1 face

$\tau < \sigma$ with $\dim(\tau) = \dim(\sigma) - 1$

- The union of the proper faces is the **boundary** of σ, $\mathrm{Bd}(\sigma)$.
- The **interior** of σ is $\sigma - \mathrm{Bd}(\sigma)$. This is sometimes called the open simplex.

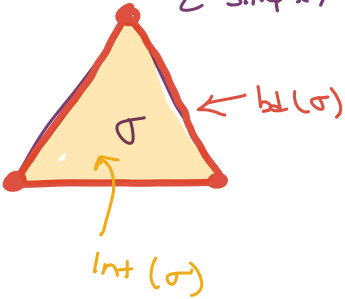
3-simplex σ



$\mathrm{Bd}(\sigma)$

$\Rightarrow$ all triangles, edges, vertices

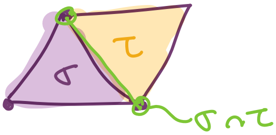
"hollow"

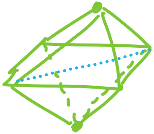


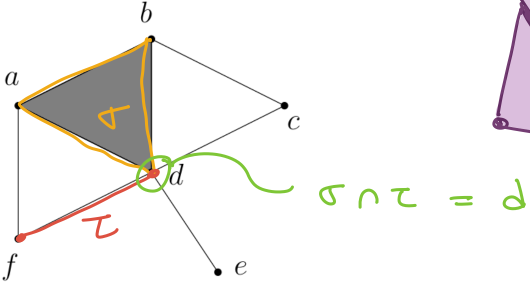
Geometric simplicial complex

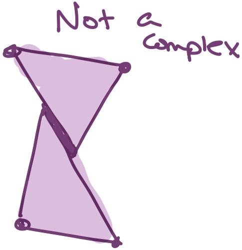
A **(geometric) simplicial complex** K in $\mathbb{R}^N$ is a (finite) collection of simplices in $\mathbb{R}^N$ such that

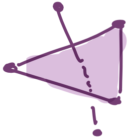
1. Every face of a simplex of K is in K.
2. The intersection of any two simplices of K is a face of each.





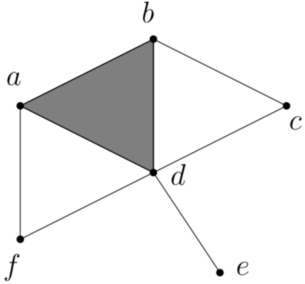






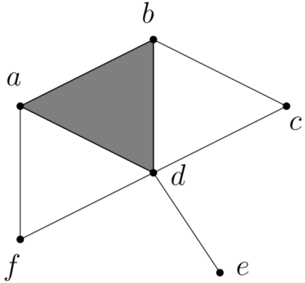
Dimension

The **dimension** of K is the maximum dimension of its simplices.



$\dim(K) = 2$

An **abstract simplicial complex** $\mathcal{K}$ is a (finite) collection of (finite) non-empty sets of V such that $\alpha \in \mathcal{K}$ and $\beta \subseteq \alpha$ implies $\beta \in \mathcal{K}$.



$V = \{a, b, c, d, e, f\}$

$K = \{ \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\},$
$\{a,b\}, \{b,c\}, \{c,d\}, \{b,d\}, \{a,d\},$
$\{f,d\}, \{a,f\}, \{d,e\},$
$\underbrace{\{a,b,d\}}_{\alpha} \}$

$\beta = \{a, d\}$ $\quad \beta \subseteq \alpha$

$K = \{ \{a\}, \{b\}, \{c\}, \{a,b,c\} \}$

Not a complex!

Two types

Geometric

Abstract

underlying space

forget the geometry

Come with embedding in $\mathbb{R}^N$

Combinatorics

List of subsets

geometric realization

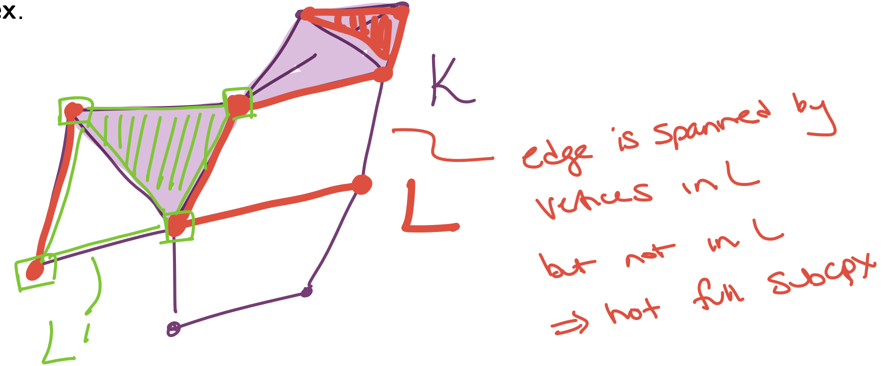
Not obvious!

$\dim(K) = N \Rightarrow$ geometric realization in $\mathbb{R}^{2N+1}$

Geometric realization, underlying space

Subcomplex

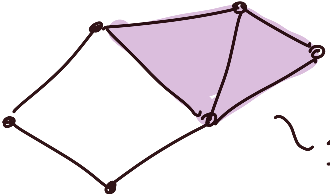
If L is a subcollection of K that contains all faces of its elements, then L is called a **subcomplex**.



A subcomplex $L \subset K$ is **full** if it contains all simplices of K spanned by the vertices in L.

Skeleton

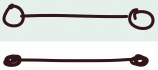
The subcomplex of K consisting of all simplices of dimension $\leq p$ is called the p-**skeleton** of K, usually denoted K^p.



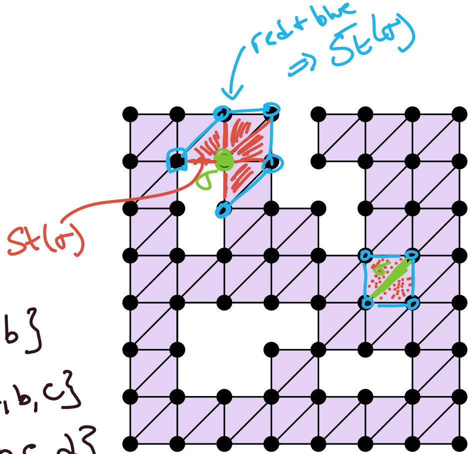
1-skeleton is the graph

Stars

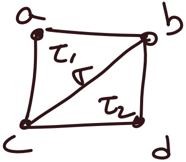
$$\mathrm{St}(\sigma) = \{ \tau \in K \mid \sigma \leq \tau \}$$



- The star of a simplex $\sigma \in K$, $\mathrm{St}(\sigma)$, is the union of the interiors of simplices in K which have σ as a face. ← geometric realization
  Warning: $\mathrm{St}(\sigma) \subset |K|$ is not a simplicial complex!
- The **closed star**, $\overline{\mathrm{St}}(\sigma)$, is the closure of $\mathrm{St}(\sigma)$.
- Link $\overline{\mathrm{St}}(\sigma) - \mathrm{St}(\sigma)$
  └ Blue



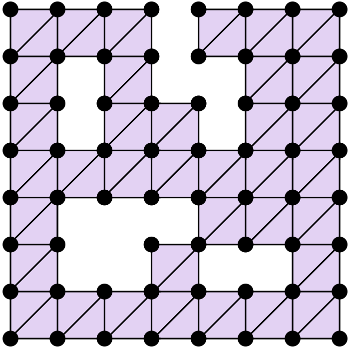
$$\mathrm{St}(\sigma) = \{\tau_1, \tau_2, \sigma\}$$



$$\sigma = \{c, b\}$$
$$\tau_1 = \{a, b, c\}$$
$$\tau_2 = \{b, c, d\}$$

Links

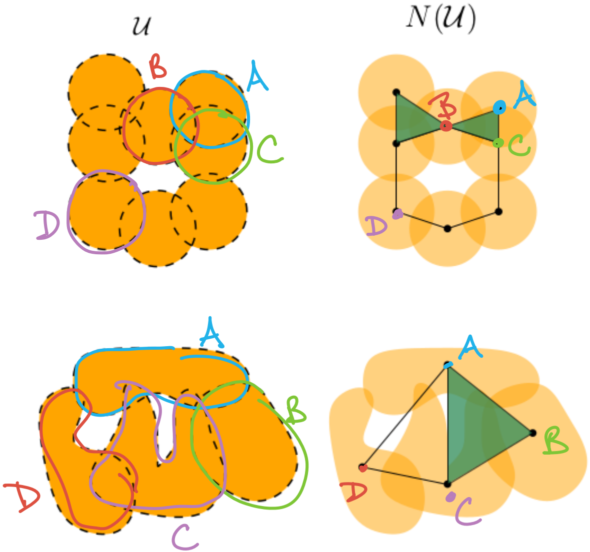
The **link** of a simplex is $\overline{\mathrm{St}}(\sigma) - \mathrm{St}(\sigma)$.



Nerve

Given a finite collection of sets $\mathcal{F}$, the **nerve** is

$$\mathrm{Nrv}(\mathcal{U}) = \{X \subseteq \mathcal{F} \mid \bigcap_{U \in X} U \neq \emptyset\}.$$



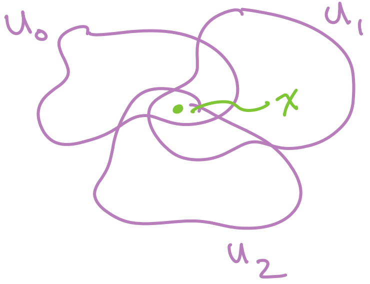
Check: This is actually an abstract simplicial complex!

Recall: K is a simplicial complex if $\sigma \in K$ and $\tau \leq \sigma$ implies $\tau \in K$.

$\sigma \in \mathrm{Nrv}(F)$

$U_0, U_1, U_2, \ldots, U_n$

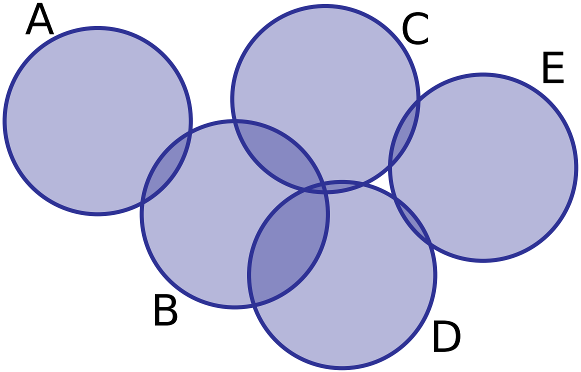
$\exists \; x \in \bigcap_{i=0}^{n} U_i$



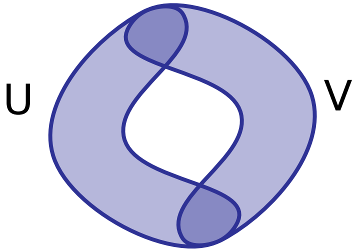
Any $\{U_0, U_1\}$ $\quad U_0 \cap U_1 \neq \emptyset$ $\quad$ (x in there too)

$$\mathrm{Nrv}(\mathcal{F}) = \{X \subseteq \mathcal{F} \mid \bigcap_{U \in X} U \neq \emptyset\}.$$

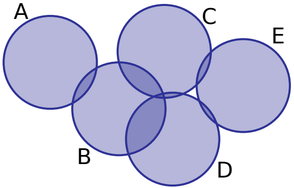
Try it: What is the nerve of this collection of disks?

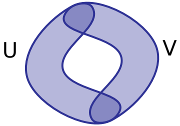


Try it: What is the nerve of this collection of sets?



The difference between the examples



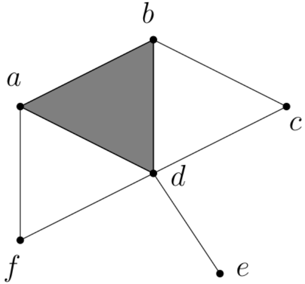


Homework

Pick one

- Let K be a simplicial complex. Show that K is the nerve of the collection of stars of its vertices.
- A flag in a simplicial complex K is a nested sequence of proper faces $\sigma_0 < \sigma_1 < \cdots < \sigma_k$. Show that the collection of flags in K forms a simplicial complex; this is called the **order complex** of K.

For either question, show the construction on the simplicial complex below.



Jared