Simplicial Complexes

Lecture 3 - CMSE 890

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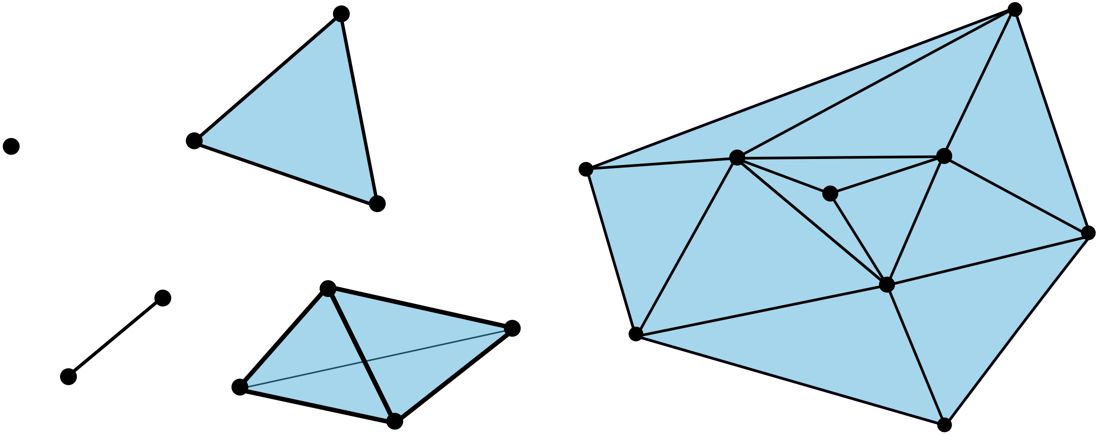
Goals

Goals for today: Ch 2.1

- Define geometric/abstract simplicial complexes
- Define simplicial maps

Idea

A simplicial complex is a generalizion of a graph



General position

A set $\{a_0, \cdots, a_n\} \subset \mathbb{R}^N$ is **in general position** if no k of them lie in a $k-2$ dimensional hyperplane ($k = 2, 3, \ldots, N+1$).

Warning: Papers will often modify this definition to suit their purposes!

n-Plane

Given a set of points $\{a_0, \cdots, a_n\}$ in general position, the n-plane P spanned by the points is

$$P = \left\{ \sum_{i=0}^{n} t_i a_i \in \mathbb{R}^N \mid \sum t_i = 1 \right\}.$$

n-simplex

Given a set of points $\{a_0, \cdots, a_n\}$ in general position, the n-simplex σ spanned by the points is

$$P = \left\{ \sum_{i=0}^{n} t_i a_i \in \mathbb{R}^N \mid \sum t_i = 1 \right\}$$

such that $t_i \geq 0 \ \forall i$.

Barycentric coordinates

The numbers t_i are uniquely determined by x and are called **barycentric coordinates**.

More definitions

- $\{a_0, \cdots, a_n\}$ are called the **vertices** of σ.
- The dimension of $\sigma = [a_0, \cdots, a_n]$ is n.
- Any simplex spanned by a subset of $\{a_0, \cdots, a_n\}$ is a **face** of σ.
- A face is proper if it isn't the simplex itself.

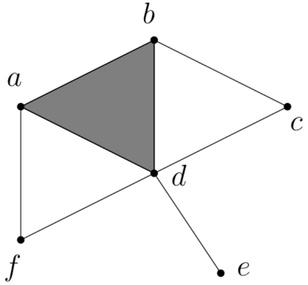
Even more definitions

- The union of the proper faces is the **boundary** of σ, $\mathrm{Bd}(\sigma)$.
- The **interior** of σ is $\sigma - \mathrm{Bd}(\sigma)$. This is sometimes called the open simplex.

Geometric simplicial complex

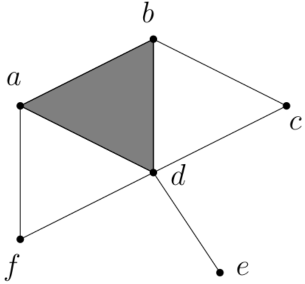
A **(geometric) simplicial complex** K in $\mathbb{R}^N$ is a (finite) collection of simplices in $\mathbb{R}^N$ such that

1. Every face of a simplex of K is in K.
2. The intersection of any two simplices of K is a face of each.



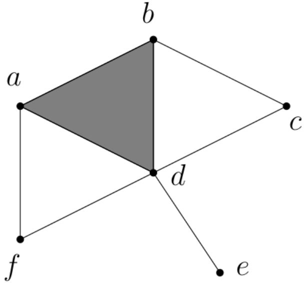
Dimension

The **dimension** of K is the maximum dimension of its simplices.



Abstract simplicial complexes

An **abstract simplicial complex** $\mathcal{K}$ is a (finite) collection of (finite) non-empty sets of V such that $\alpha \in \mathcal{K}$ and $\beta \subseteq \alpha$ implies $\beta \in \mathcal{K}$.



Two types

Geometric

Abstract

Geometric realization, underlying space

Subcomplex

If L is a subcollection of K that contains all faces of its elements, then L is called a **subcomplex**.

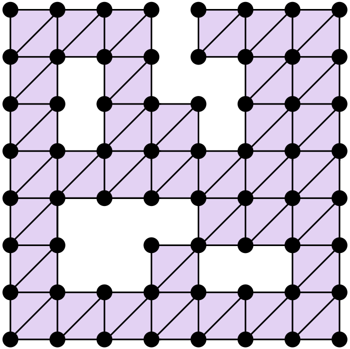
A subcomplex $L \subset K$ is **full** if it contains all simplices of K spanned by the vertices in L.

Skeleton

The subcomplex of K consisting of all simplices of dimension $\leq p$ is called the p-**skeleton** of K, usually denoted K^p.

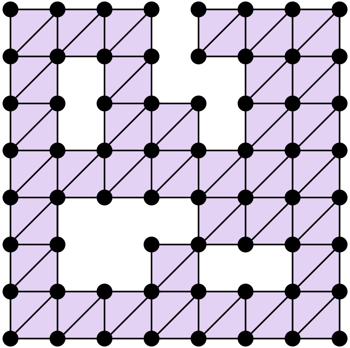
Stars

- The star of a simplex $\sigma \in K$, is the set of simplices that have σ as a face:
  $\mathrm{St}(\sigma) = \{\tau \in K \mid \sigma \leq \tau\}$.
  Warning: $\mathrm{St}(\sigma) \subset |K|$ is not a simplicial complex!
- The **closed star**, $\overline{\mathrm{St}}(\sigma)$, is the closure of $\mathrm{St}(\sigma)$.



Links

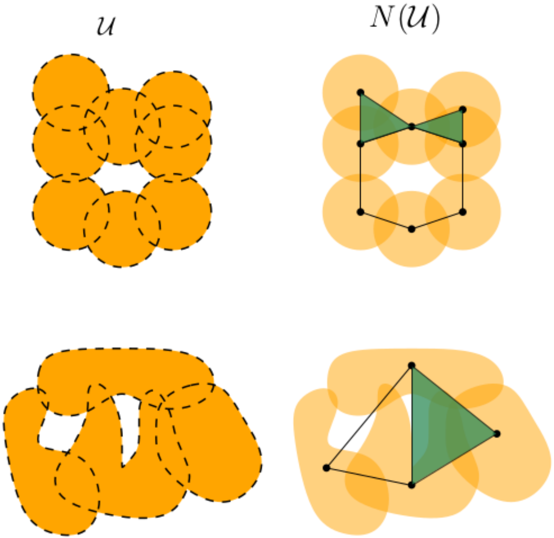
The **link** of a simplex is $\overline{\mathrm{St}}(\sigma) - \mathrm{St}(\sigma)$.



Nerve

Given a finite collection of sets $\mathcal{F}$, the **nerve** is

$$\mathrm{Nrv}(\mathcal{U}) = \{X \subseteq \mathcal{F} \mid \bigcap_{U \in X} U \neq \emptyset\}.$$

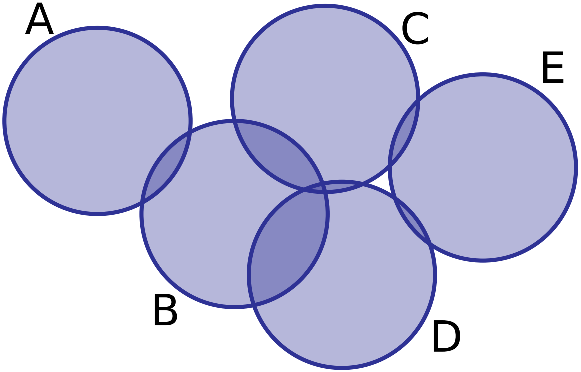


Check: This is actually an abstract simplicial complex!

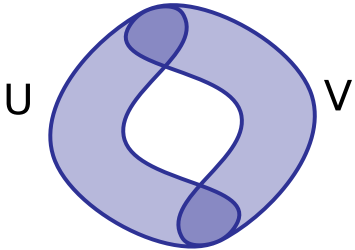
Recall: K is a simplicial complex if $\sigma \in K$ and $\tau \leq \sigma$ implies $\tau \in K$.

$$\mathrm{Nrv}(\mathcal{F}) = \{X \subseteq \mathcal{F} \mid \bigcap_{U \in X} U \neq \emptyset\}.$$

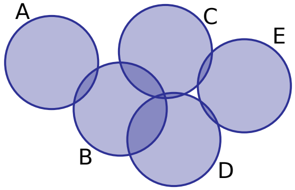
Try it: What is the nerve of this collection of disks?

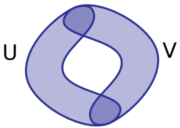


Try it: What is the nerve of this collection of sets?



The difference between the examples





Homework

Pick one

- Let K be a simplicial complex. Show that K is the nerve of the collection of stars of its vertices.
- A flag in a simplicial complex K is a nested sequence of proper faces $\sigma_0 < \sigma_1 < \cdots < \sigma_k$. Show that the collection of flags in K forms a simplicial complex; this is called the **order complex** of K.

For either question, show the construction on the simplicial complex below.