PL Functions and Reeb Graphs

Lecture 16 - CMSE 890

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Check-in

Goals for today:

- 3.1 Filtrations on spaces, PL Functions
- 3.5 Persistence for PL Functions

7.1 Reeb graphs

Section 1

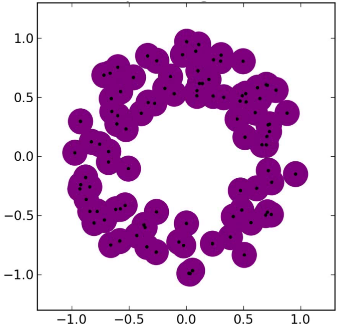
Persistence on functions

A different viewpoint for point cloud persistence

Given point cloud $P \subset \mathbb{R}^d$.

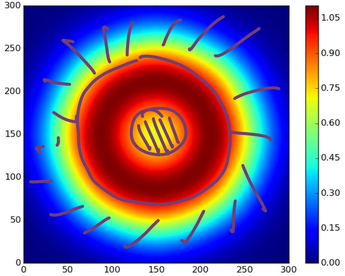
$$f : \mathbb{R}^d \to \mathbb{R}$$
$$x \mapsto \min_{y \in P} \|x - y\|$$

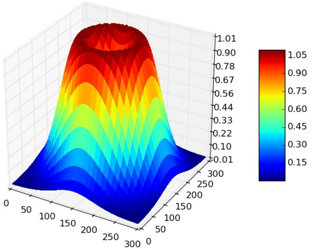
$$f^{-1}(-\infty, r] = \bigcup_{y \in P} B_r(y)$$



Different functions

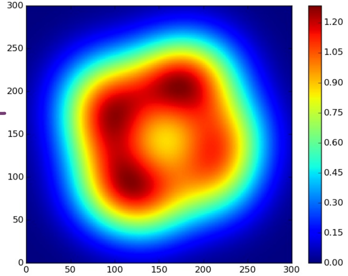
$f_1 : \mathbb{R}^2 \to \mathbb{R}$

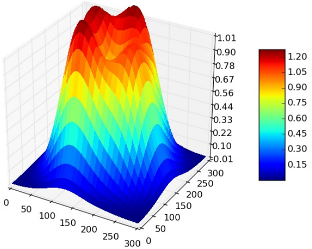




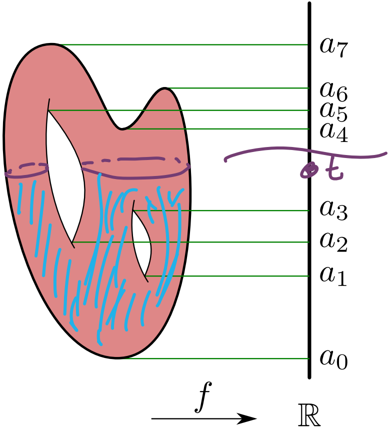
$f^{-1}(-\infty, 0.75)$

$f_2 : \mathbb{R}^2 \to \mathbb{R}$





Another example: “Height functions”



$f^{-1}(t)$

$f^{-1}(-\infty, t]$

Persistence for functions

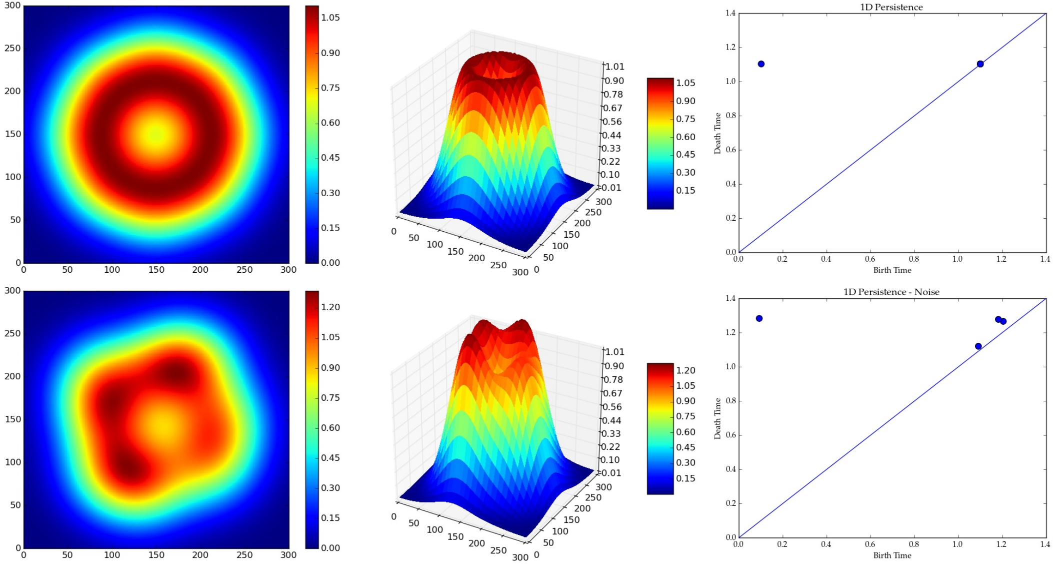
Let $\mathbb{X}$ be a topological space with function $f : \mathbb{X} \to \mathbb{R}$.
Let $\mathbb{X}_a = f^{-1}(-\infty, a]$.
Then for $a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_k$

$$\mathbb{X}_{a_0} \subseteq \mathbb{X}_{a_1} \subseteq \cdots \subseteq \mathbb{X}_{a_k}$$

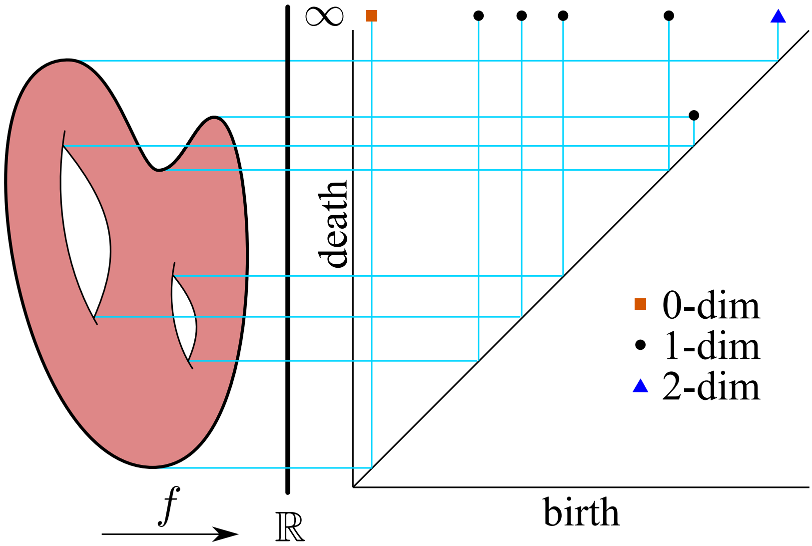
gives

$$H_p(\mathbb{X}_{a_0}) \to H_p(\mathbb{X}_{a_1}) \to \cdots \to H_p(\mathbb{X}_{a_k}).$$

Example with matrix function



Example with torus height function



Section 2

Piecewise Linear functions

PL Functions

Given simplicial complex K. A function

$$f : V(K) \to \mathbb{R}$$

gives a piecewise linear (PL) function

$$f : |K| \to \mathbb{R}$$

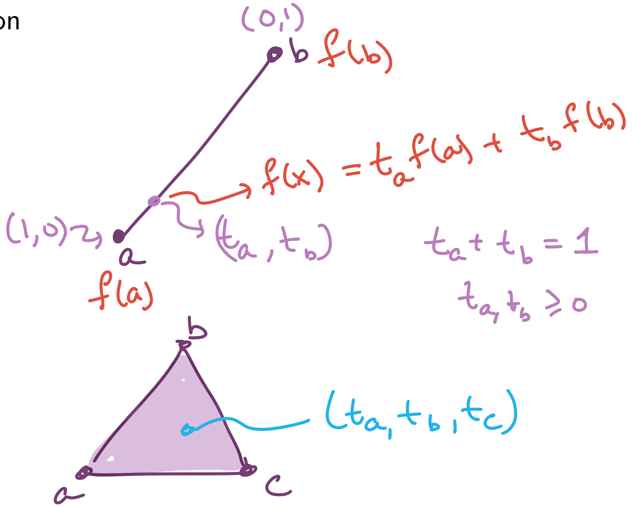
$$x \mapsto \sum_{v \in V(K)} (b_i(x) f(v)).$$

The function is *generic* if

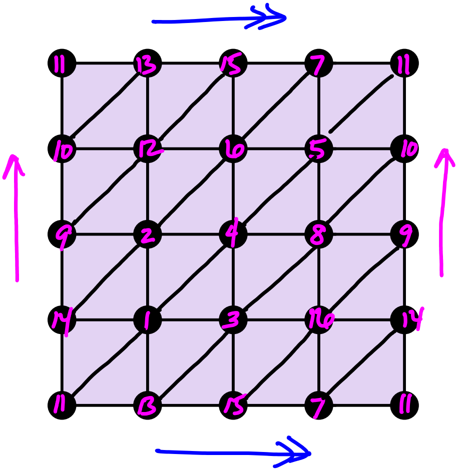
$$f(u) \neq f(v) \; \forall u \neq v \in V(K).$$

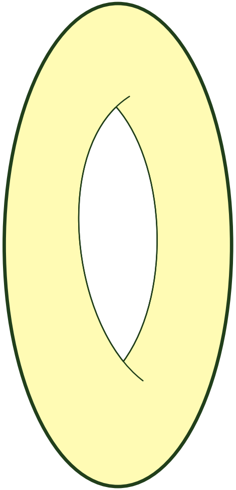
Sort the vertices so that

$$f(u_1) < f(u_2) < \cdots < f(u_n).$$

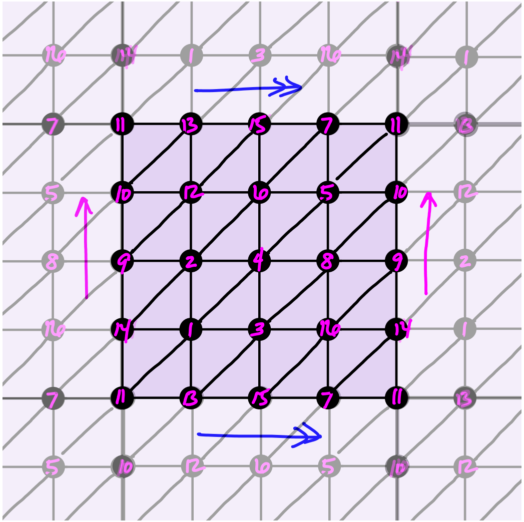


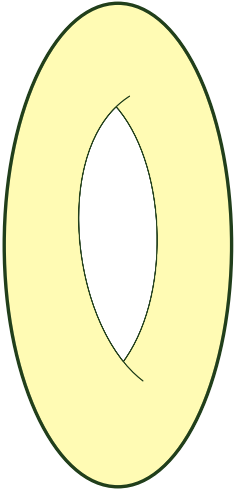
Square representation of a torus





Square representation of a torus





Level set

Given (PL) $f : |K| \to \mathbb{R}$.
A sublevel set:

$$f^{-1}(-\infty, a]$$

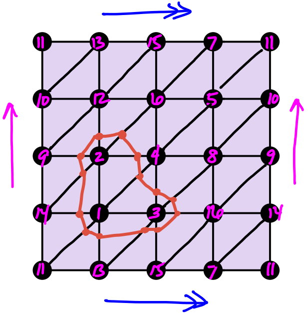
A superlevel set:

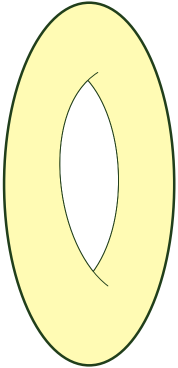
$$f^{-1}[a, \infty)$$

A level set:

$$f^{-1}(a)$$

$f^{-1}(3.3)$

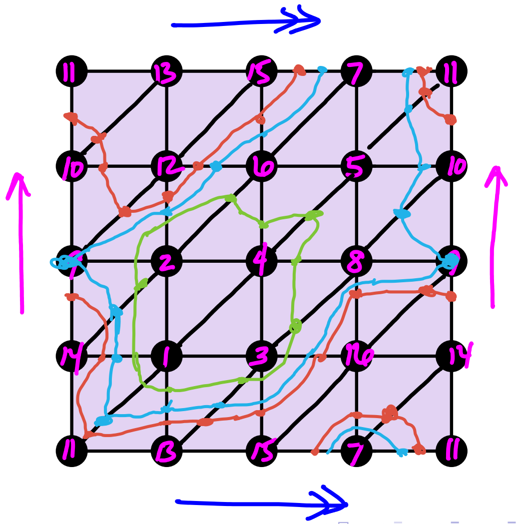




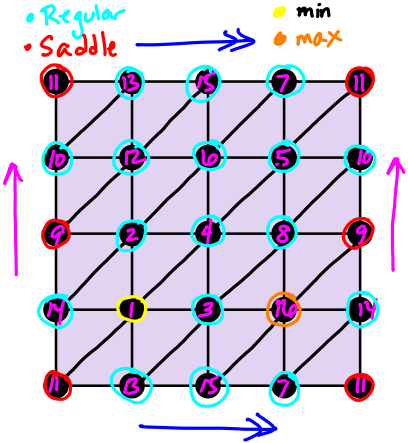
Try it:

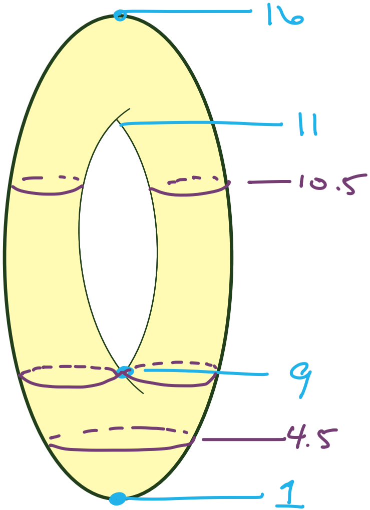
Draw the following levelsets:

- $f^{-1}(4.5)$
- $f^{-1}(10.5)$
- $f^{-1}(9)$



Matching this example with the height function





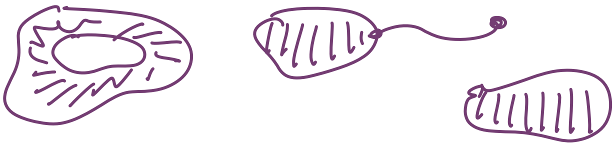
Section 3

Reeb graphs

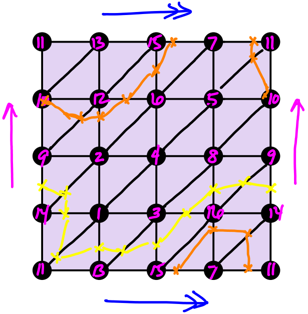
Recall: Connected Components

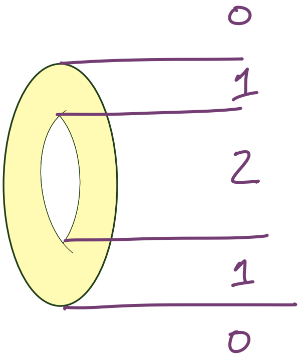
A space X is disconnected if there are disjoint open sets U and V with $X = U \cup V$. A space is connected otherwise.

A connected component is a maximal subset that is connected.



A contour is a connected component of a level set.

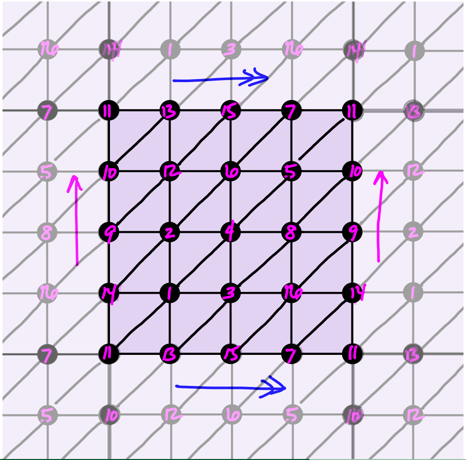




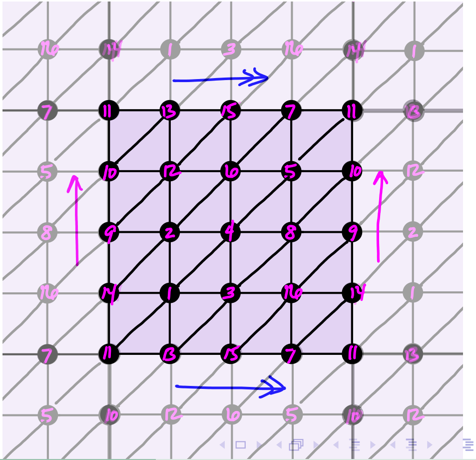
How do the number of contours change as the level set function value changes?

Try it

Sketch the level set $f^{-1}(8)$. What shape is it? How many connected components does it have?



Sketch the level set $f^{-1}(9)$. What shape is it? How many connected components does it have?



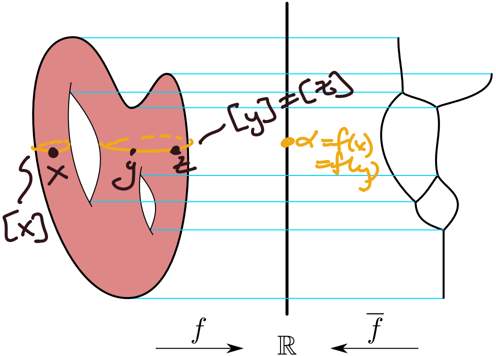
Equivalence relation

A binary relation $\sim$ on a set $\mathbb{T}$ is an equivalence relation if for every $a, b, c \in \mathbb{T}$:

- $a \sim a$ (reflexivity)
- $a \sim b$ iff $b \sim a$ (symmetry)
- If $a \sim b$ and $b \sim c$ then $a \sim c$

The equivalence class of a under $\sim$ is $[a] = \{x \in \mathbb{T} \mid x \sim a\}$.

Reeb graph equivalence relation



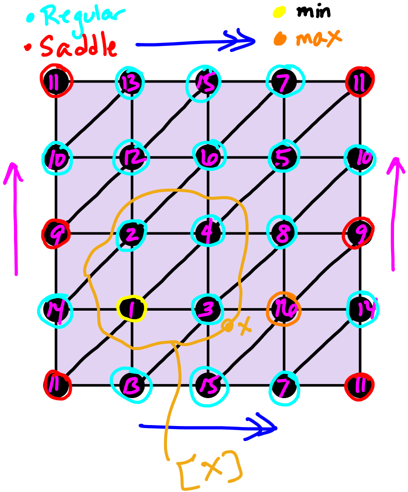
Given a function $f : X \to \mathbb{R}$.
Define an equivalence relation $\sim$ by
$x \sim y$ iff

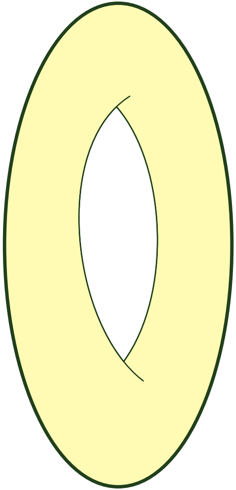
- $f(x) = f(y) = \alpha$
- x and y are in the same connected component of the level set $f^{-1}(\alpha)$.

$y \sim z$

$x \nsim y$ or z

Equivalence relation example





Quotient space

Given a topological space $(\mathbb{T}, T)$ and an equivalence relation $\sim$ defined on the set $\mathbb{T}$, the quotient space $(\mathbb{S}, S)$ consists of

- points $\mathbb{S} = \mathbb{T}/\sim = \{[a] \mid a \in \mathbb{T}\}$
- open sets $S = \{U \subseteq \mathbb{S} \mid \{a \mid [a] \in U\} \in T\}$

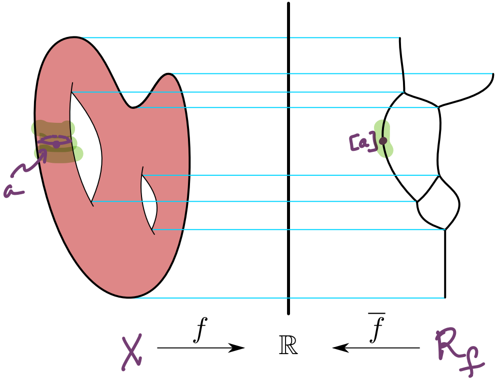
$U = \{[a], [b], \ldots \ldots\}$

$a_1, a_2, a_3 \ldots\ldots$

V.2:

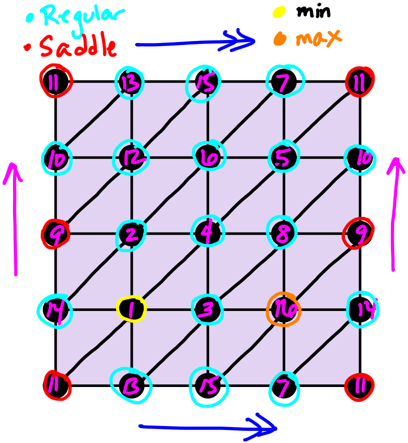
- Get a map $q : \mathbb{T} \to \mathbb{S}$, $a \mapsto [a]$
- $U \subseteq \mathbb{S}$ open iff $q^{-1}(U)$ is open

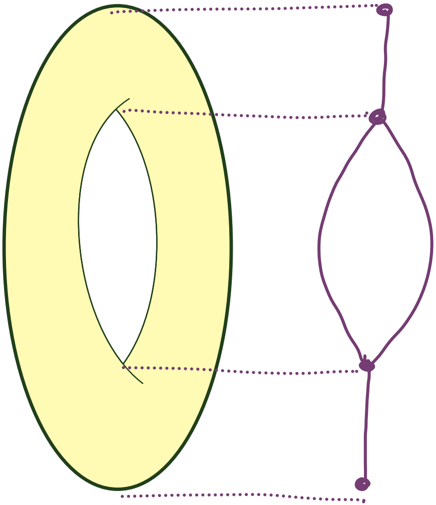
Reeb graph



- Let $[x]$ denote the equivalence class of $x \in X$.
- The Reeb graph R_f of $f : X \to \mathbb{R}$ is the quotient space $X/\sim$.
- Let $\Phi : X \to R_f$, $x \to [x]$ be the quotient map.

Computing the Reeb graph





Nice enough assumptions

If the input is nice enough, then R_f is a graph.....

- Morse function on a compact manifold.
- PL function on a compact polyhedron
- Others......

geometric graph

1-dimensional stratified

Their assumptions

Require that $f : X \to \mathbb{R}$ is levelset tame:

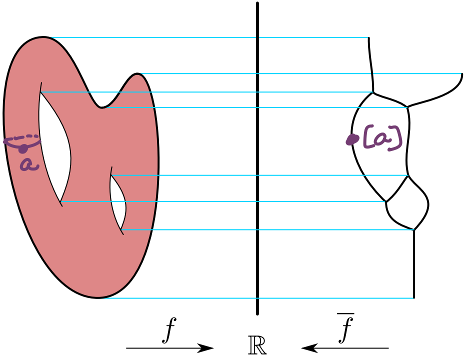
- each level set $f^{-1}(a)$ has a finite number of components
- each component is path connected
- f is of Morse type *(Defn 4.14: each levelset has finitely generated homology groups, finitely many critical vluaes, each interval between critical values is regular)*

Reeb graph as a topological summary

With a bit of a lie

Figure: Pascucci et al

Induced map $\tilde{f} : R_f \to \mathbb{R}$



Lazy notation: Just write $f : R_f \to \mathbb{R}$

Another example

Compute the Reeb graph of the following example: