Bottleneck Distance and Stability

Lecture 13 - CMSE 890

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Goals

- Bottleneck distance and stability
- Wasserstein distance

Section 1

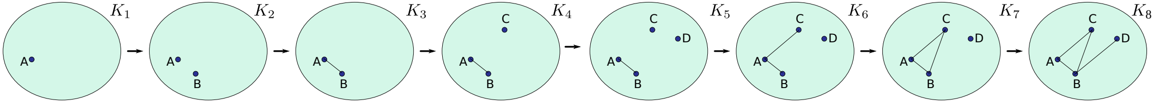
Review of setup

Simplicial Complex Filtration from a Function

Definition

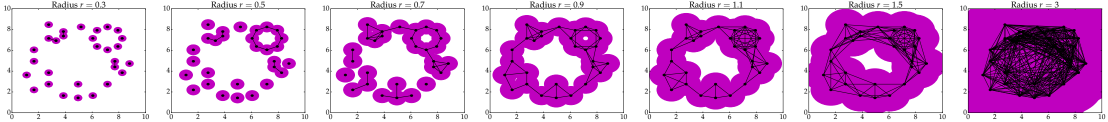
Fix a function $f : K \to \mathbb{R}$ with $(\sigma \leq \tau \Rightarrow f(\sigma) \leq f(\tau))$
Fix values $a_1 \leq a_2 \leq \cdots \leq a_n$
Let $K_i = f^{-1}(-\infty, a_i]$
A filtration $\mathcal{F} = \mathcal{F}(K)$ induced by f is a nested sequence of subcomplexes

$$\mathcal{F} : \emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K.$$

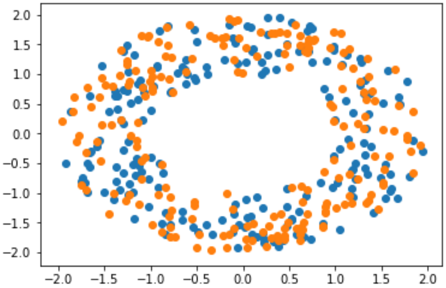


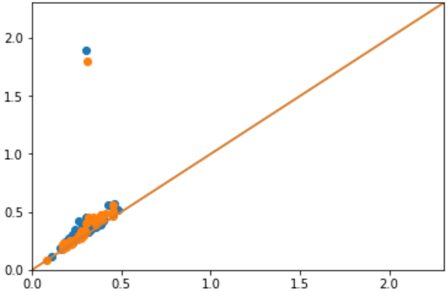
Rips filtration

- Given $P \subseteq \mathbb{R}^d$
- Fix $0 \leq r_1 \leq r_2 \cdots \leq r_n$
- Set $K_i = VR(P, r_i)$
- Compute persistence diagram using these radius parameters



Similar persistence diagrams





Section 2

Bottleneck distance

Distance measures

Definition

A distance on a set M is a function $d : M \times M \to \mathbb{R}_{\geq 0}$ such that

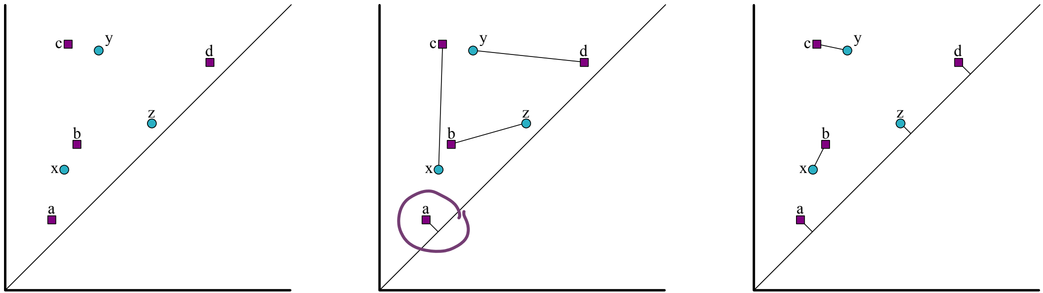
- $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

Ex. $\|(x_1, x_2) - (y_1, y_2)\|_2 = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$

Ex. $\|(x_1, x_2) - (y_1, y_2)\|_\infty = \max\{|x_1-y_1|, |x_2-y_2|\}$

Finding matchings

$d(\mathrm{Dgm}_1, \mathrm{Dgm}_2)$

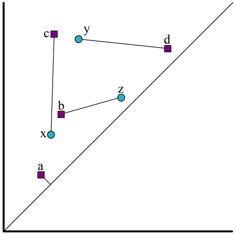


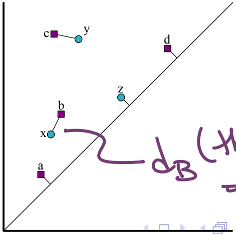
Definition of bottleneck, Version 1

Given diagrams X and Y, the distance between them is

$$d_B(X, Y) = \inf_{\underbrace{\varphi : X \to Y}_{\text{isomorphism}}} \sup_{x \in X} \|x - \varphi(x)\|_\infty$$

there are countably many copies of every point on the diagonal



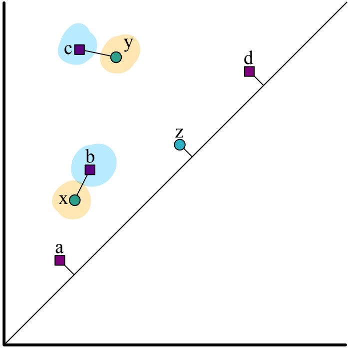


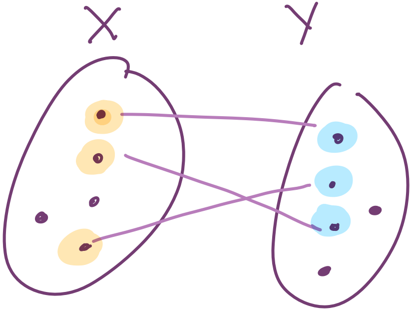
$d_B(\text{these}) = \|x - b\|_\infty$

Definition of bottleneck, Version 2

Matching

A partial matching between X and Y is a bijection on a subset of the off-diagonal points of the two diagrams $M : X' \to Y'$ for $X' \subseteq X$ and $Y' \subseteq Y$.



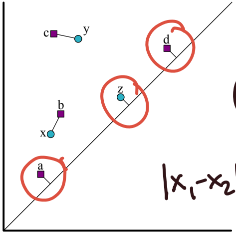


Definition of bottleneck, Version 2

Cost of a matching

The cost of this partial matching is given by

$$c(M) = \sup \left(\{ \|x - M(x)\|_\infty \mid x \in X' \} \cup \{ \overbrace{\tfrac{1}{2}|x_1 - x_2|}^{\|x - \Delta\|} \mid (x_1, x_2) \in \underbrace{X \setminus X' \cup Y \setminus Y'}_{\text{unmatched points in both dgms}} \} \right)$$



(x_1, x_2)

$|x_1 - x_2|$

$\left(\tfrac{1}{2}|x_1 - x_2|, \tfrac{1}{2}|x_1 - x_2|\right)$

Bottleneck distance

Combinatorial definition

The bottleneck distance between X and Y is

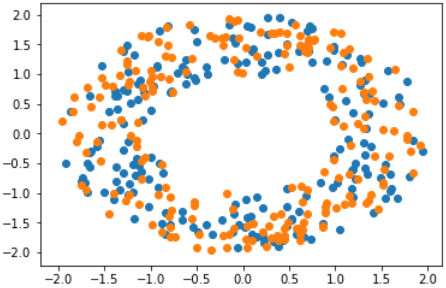
$$d_B(X, Y) = \inf_M c(M)$$

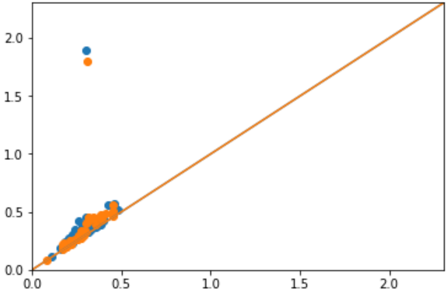
where the infimum runs over all possible partial matchings.

Section 3

Bottleneck stability

Similar persistence diagrams





Stability - Point cloud version

Theorem (Point cloud version)

Let $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^d$ be a finite point clouds. Let d_H be Hausdorff distance. Let $D(\mathcal{C}(\mathbb{X}))$ be the persistence diagram of the filtration defined by the Čech complex on $\mathbb{X}$. Then

$$0 \leq d_B(\mathrm{Dgm}_p(\mathcal{C}(\mathbb{X})), \mathrm{Dgm}_p(\mathcal{C}(\mathbb{Y}))) \leq \underbrace{d_H(\mathbb{X}, \mathbb{Y})}_{\text{Small}}.$$

Stability - Function version

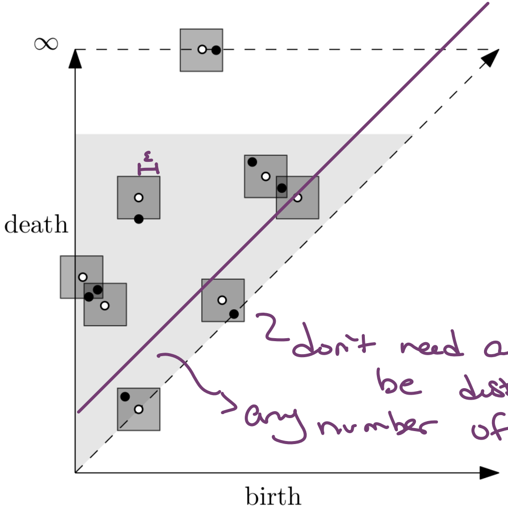
Theorem (Stability for simplicial functions)

Let $f, g : K \to \mathbb{R}$ be two simplex-wise monotone functions giving rise to two simplicial filtrations $\mathcal{F}_f$ and $\mathcal{F}_g$.
Then, for every $p \geq 0$,

$$d_B(\mathrm{Dgm}_p(\mathcal{F}_f), \mathrm{Dgm}_p(\mathcal{F}_g)) \leq \underbrace{\|f - g\|_\infty}.$$

$$= \max_{\sigma \in K} |f(\sigma) - g(\sigma)|$$

Nearby diagrams



Start w/ white circ dgm

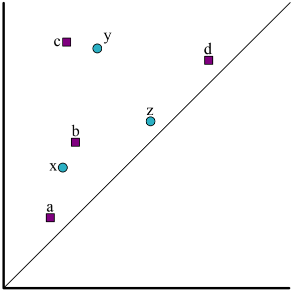
Fix $\varepsilon > 0$

What diagrams are within distance ε of this dgm?

2 don't need a pt to be dist ε

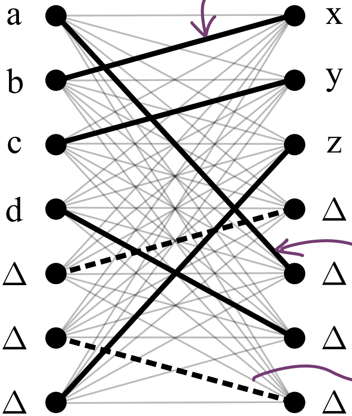
any number of extra pts here

Computing Bottleneck distance



□ dgm

cost of edge $\|b-x\|_\infty$



○ circle dgm

$\|a-\Delta\|_\infty$

0

Section 4

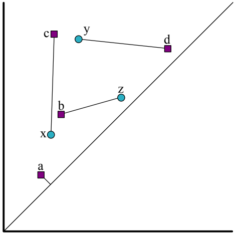
Wasserstein Distance and Stability

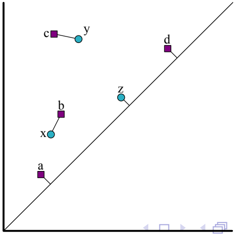
Wasserstein Distance

Given diagrams X and Y, the p-Wasserstein distance between them is

$$d_{W,p}(X,Y) = \inf_{\varphi : X \to Y} \left(\sum_{x \in X} \|x - \varphi(x)\|_p^p \right)^{1/p}$$

$$\left(\|x - \varphi(x)\|_p \right)^p$$





Wasserstein Stability

Theorem (Stability for q-Wasserstein distance)

Let $f, g : K \to \mathbb{R}$ be two simplex-wise monotone functions on a simplicial complex K. Then,

$$d_{W,q}(\mathrm{Dgm}_p(\mathcal{F}_f), \mathrm{Dgm}_p(\mathcal{F}_g)) \leq \|f - g\|_q.$$

dim
of
pers
hom

What's the difference?

...To the jupyter notebook!!!