Here Comes the Homology

Are we actually going to get there?
Lecture 6 - CMSE 890

Prof. Elizabeth Munch

Michigan State University
::
Dept of Computational Mathematics, Science & Engineering

Thurs, Sep 11, 2025

Goals

Goals for today:

- Homology!

Section 1

More on the boundary map

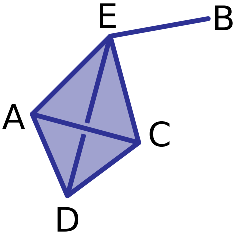
p-Chains

Let K be a simplicial complex and fix a dimension p.

- A p-chain is a formal sum of p-simplices in K, written

$$\alpha = \sum a_i \sigma_i$$

- p-chains are added component-wise: if $\alpha = \sum a_i \sigma_i$ and $\beta = \sum b_i \sigma_i$, then $\alpha + \beta = \sum (a_i + b_i)\sigma_i$
- The collection of p-chains with addition is called the p^{th}-chain group (vector space), $C_p(K)$.

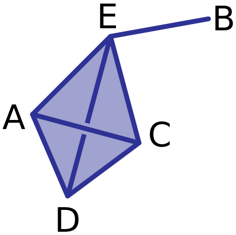


Boundary maps[1]

$$\begin{array}{rccc} \partial_p : & C_p(K) & \to & C_{p-1}(K) \\ & \sigma = [v_0, \cdots, v_p] & \mapsto & \sum_{j=0}^{p} [v_0, \cdots, \widehat{v_j}, \cdots v_p] \end{array}$$

[1] Warning: We are assuming $\mathbb{Z}_2$ coefficients from now on!

Matrix representation



$$\partial_1(K) = \begin{array}{cc} & \begin{array}{ccccccc} AC & AD & AE & BE & CD & CE & DE \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{pmatrix} & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{pmatrix} \end{array}$$

Chain complex

$$\cdots \xrightarrow{\partial_{p+2}} C_{p+1}(X) \xrightarrow{\partial_{p+1}} C_p(X) \xrightarrow{\partial_p} C_{p-1}(X) \xrightarrow{\partial_{p-1}} \cdots$$

$$\begin{array}{rccc} \partial_p : & C_p(K) & \to & C_{p-1}(K) \\ & \sigma = [v_0, \cdots, v_p] & \mapsto & \sum_{j=0}^{p} [v_0, \cdots, \widehat{v_j}, \cdots v_p] \end{array}$$

Section 2

Cycles and Boundaries

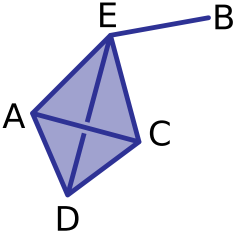
Important subspaces for a linear transformation

- Image

- Kernel

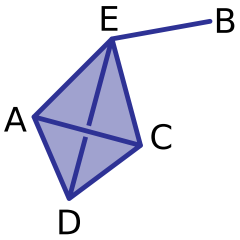
Cycles

A chain in the kernel of ∂_p is called a p-**cycle**. $C_{p+1}(K) \xrightarrow{\partial_{p+1}} C_p(K) \xrightarrow{\partial_p} C_{p-1}(K)$

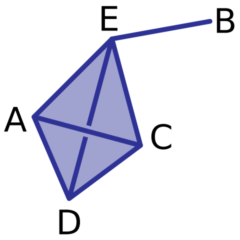


The collection of p-cycles forms a subspace $Z_p(K) \subseteq C_p(K)$.

What is a 2-cycle?

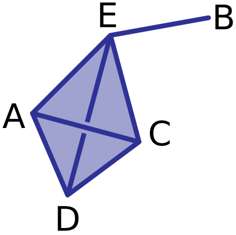


More work space if needed



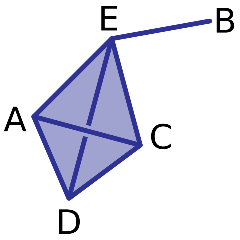
Boundaries

A chain in the image of ∂_{p+1} is called a p-**boundary**. $C_{p+1}(K) \xrightarrow{\partial_{p+1}} C_p(K) \xrightarrow{\partial_p} C_{p-1}(K)$



The collection of p-boundaries forms a subspace $B_p(K) \subseteq C_p(K)$.

More work space



Nifty trick

Theorem

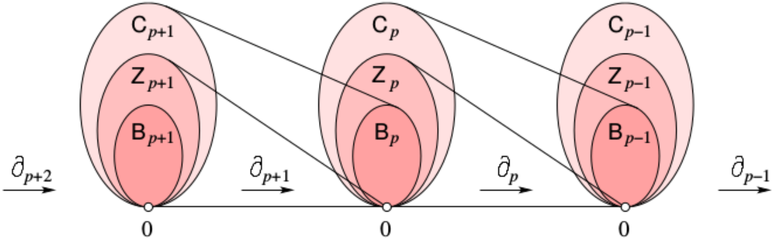
$\partial_p \partial_{p+1}(\alpha) = 0$ *for every* $(p+1)$*-chain* α.

Translation

Every p-boundary is a p-cycle.

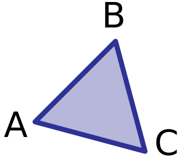
$B_p(K) \subseteq Z_p(K) \subseteq C_p(K)$

$$C_{p+1}(K) \xrightarrow{\partial_{p+1}} C_p(K) \xrightarrow{\partial_p} C_{p-1}(K)$$



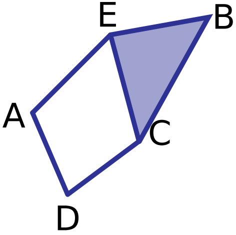
Try it: Cycles and boundaries

What are the generators of $B_1(K)$? Of $Z_1(K)$?



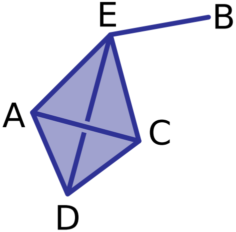
Try it: Cycles and boundaries

What are the generators of $B_1(K)$? Of $Z_1(K)$?



Try it: Cycles and boundaries

What are the generators of $B_1(K)$? Of $Z_1(K)$?



Homework (In case we only get this far)

- DW 2.6.3) Let K be the simplicial complex of a tetrahedron. Write a basis for the chain groups C_1 and C_2; boundary groups B_1 and B_2; and cycle groups Z_1 and Z_2. Write the boundary matrix representing the boundary operator ∂_2 with rows and columns representing bases of C_1 and C_2 respectively.

Section 3

Homology for real now

Quotient space

Let V be a vector space over a field k.
Let $W \subset V$ be a subspace.
Define $\sim$ on V by $x \sim y$ iff $x - y \in W$.
The equivalence class of x is denoted

$$[x] = x + W = \{x + w : w \in W\}.$$

The quotient space V/W is then defined as $\{[x] \mid x \in V\}$. This is also a vector space with:

- Scalar multiplication:
- Addition:

Homology

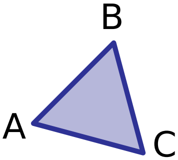
Definition

The p^{th} homology group is the quotient space

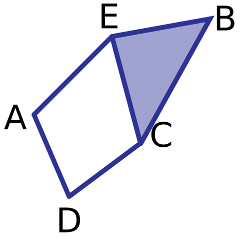
$$H_p(K) := Z_p(K)/B_p(K)$$

Spare blank page

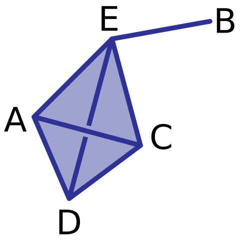
Tryit: What is $H_1(K)$?



Tryit: What is $H_1(K)$?



Tryit: What is $H_2(K)$?



Homework

- Almost certainly didn't finish all the examples above....