The Persistent Homology Transform and Monodromy

Lecture 21 - CMSE 890

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Thurs, Nov 20, 2025

Plan

- The Persistent Homology Transform (PHT)
- Monodromy

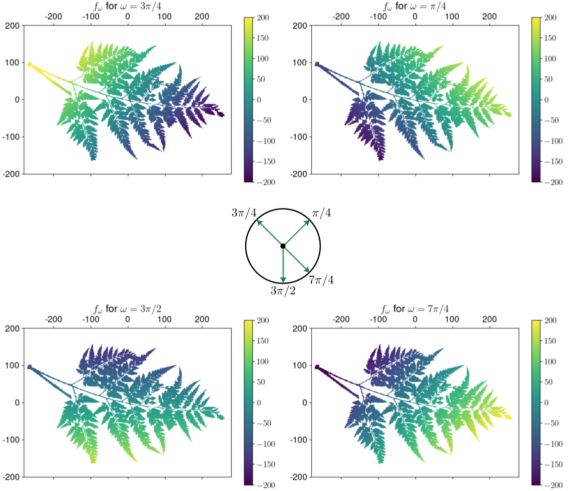
Section 1

From Last Time: PHT

Direction induces a function

d=2

- Fix a subset of $\mathbb{R}^d$, A
- Fix a direction $\omega \in \mathbb{S}^{d-1}$ = $\mathbb{S}^1$
- $f_\omega : A \to \mathbb{R}$, $f_\omega(x) = \langle x, \omega \rangle$



Function induces an algebraic representation

- χ_ω = Euler characteristic curve of f_ω
- $Pers_{k,\omega}$ = k-dimensional persistence diagram of f_ω

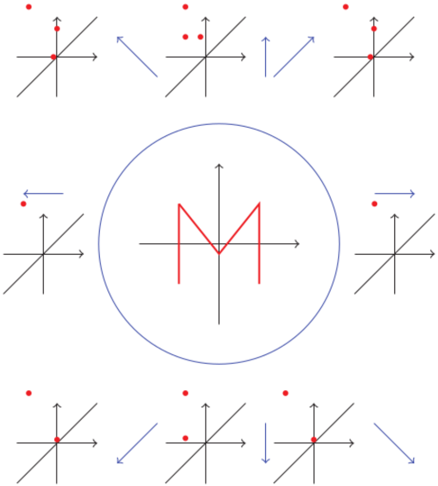


Figure from: Turner, Mukherjee, Boyer. Persistent homology transform for modeling shapes and surfaces.

The full transform(s)

$$\begin{aligned} PHT(A): \quad \mathbb{S}^{d-1} &\longrightarrow \mathcal{D}gm^d \\ \omega &\longmapsto (Pers_{0,\omega}(A), \ldots, Pers_{d-1,\omega}(A)) \end{aligned}$$

$$\begin{aligned} ECT(A): \quad \mathbb{S}^{d-1} &\longrightarrow \text{Functions on } \mathbb{R} \\ \omega &\longmapsto \chi_\omega(A) \end{aligned}$$

Meta transforms

$\mathcal{M}_d$ = space of all finite simplicial complexes embedded in $\mathbb{R}^d$

$$\begin{array}{rccc} PHT: & \mathcal{M}_d & \longrightarrow & \text{Functions from } \mathbb{S}^{d-1} \text{ to } \mathcal{D}gm^d \\ & A & \longmapsto & \omega \mapsto (Pers_{0,\omega}(A), \ldots, Pers_{d-1,\omega}(A)) \end{array}$$

$$\begin{array}{rccc} ECT: & \mathcal{M}_d & \longrightarrow & \text{Functions from } \mathbb{S}^{d-1} \text{ to (Functions on } \mathbb{R}) \\ & A & \longmapsto & \omega \mapsto \chi_\omega(A) \end{array}$$

Injectivity theorem

Theorem (Turner, Mukherjee, Boyer)

The ECT and PHT are injective on the space $\mathcal{M}_d$ of finite simplicial complexes embedded in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Generalizations by Ghrist et al; Curry et al

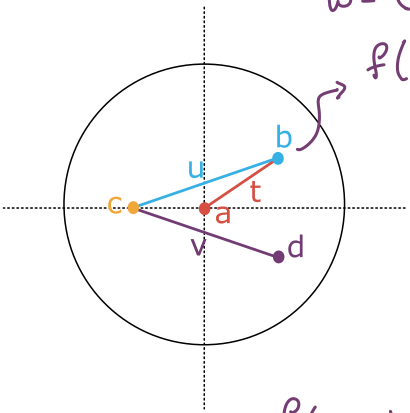
Section 2

A smaller example

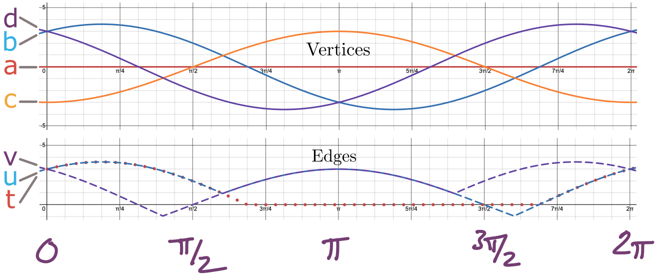
Input space

$\omega = (0,1) \in S^1$

$f(\omega, b) = \langle \omega, x_b \rangle = x_{b,1}\cos(\omega) + x_{b,2}\sin\theta$



$f(\cdot, \sigma)$ for each $\sigma \in K$

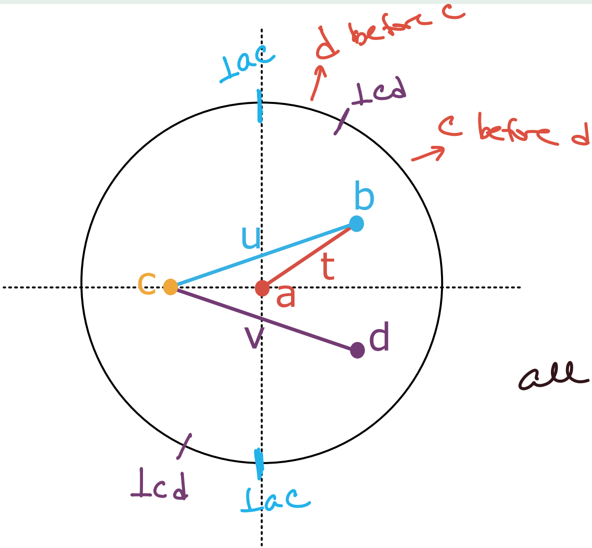


$f(\omega, u) = \max\{ f(\omega, c), f(\omega, b) \}$

https://www.desmos.com/calculator/ft1jbnlysm

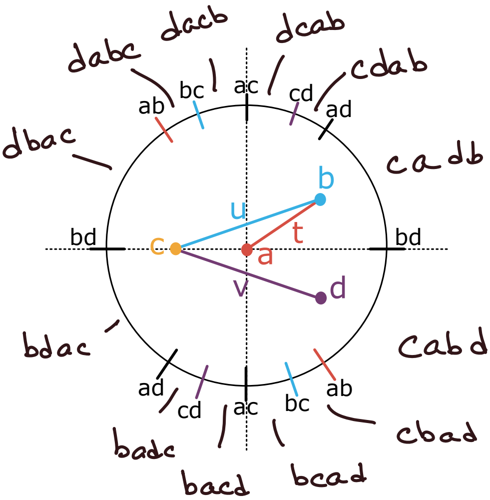
Breaking up $\mathbb{S}^1$

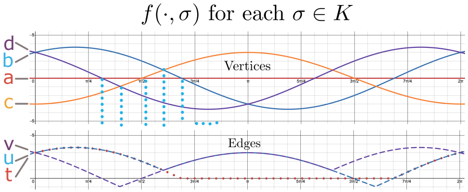
Where in $\mathbb{S}^1$ do vertices swap order?



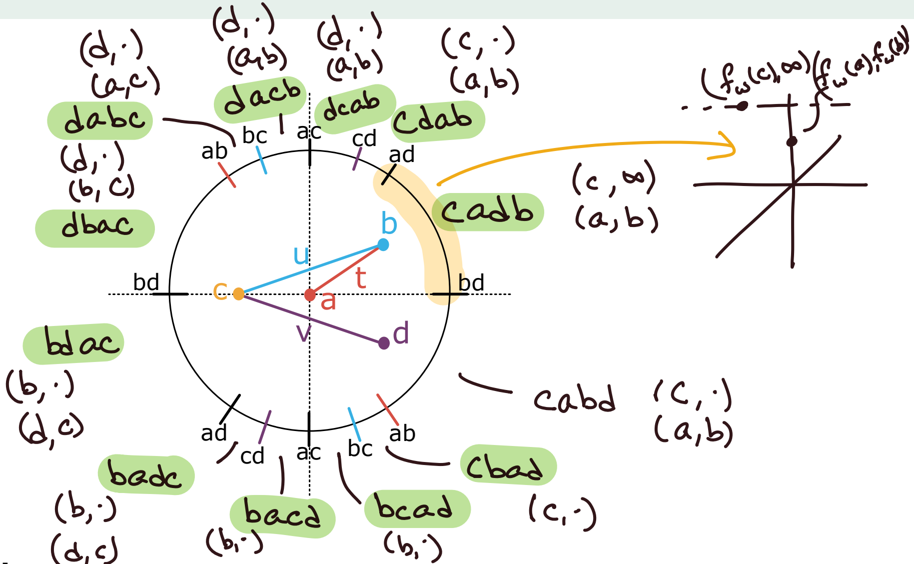
all pairs ....

Order of vertices inside regions





Persistence diagrams



Example for $[0, \sim \pi/4]$: `https://www.desmos.com/calculator/xzqm2g98p6`

Section 3

Finitely Many Directions

The problem of needing all directions

$\mathcal{M}_d =$ space of all finite simplicial complexes embedded in $\mathbb{R}^d$

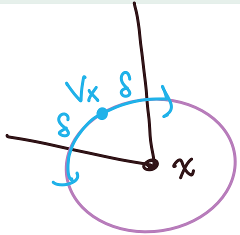
$$
\begin{array}{rccc}
PHT: & \mathcal{M}_d & \longrightarrow & \text{Functions from } \mathbb{S}^{d-1} \text{ to } \mathcal{D}gm^d \\
 & A & \longmapsto & \omega \mapsto (Pers_{0,\omega}(A), \ldots, Pers_{d-1,\omega}(A))
\end{array}
$$

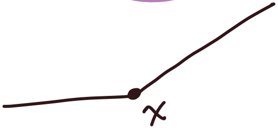
Controlling the complexes

Definition

Let $\mathcal{K}(d, \delta, k_\delta)$ be the set of embedded simplicial complexes K in $\mathbb{R}^d$ such that:

- Every vertex $x \in K$ is at least δ-observable.
  - $\exists$ a ball of directions $B(v_x, \delta) \subseteq \mathbb{S}^d$ such that the homology changes at $\langle v, x \rangle$ for all $v \in B(v_x, \delta)$.
- For all $v \in \mathbb{S}^{d-1}$, the number of $x \in X$ for which $h_w(x)$ is a homological critical value of h_w for some $w \in B(v, \delta)$ is bounded by k_δ.
  - k_δ is a global bound on the number of vertices that can change the homology for any ball $B(v, \delta)$.





From: Curry, Mukherjee, Turner. How many directions determine a shape and other sufficiency results for two topological transforms. 2022.

Finitely many directions

ex $k_\delta = 4$
$d = 2$
$\delta = 1$?

exponential in dimension

curse of dimensionality

Theorem (Curry, Mukherjee, Turner 2022)

Any shape in $\mathcal{K}(d, \delta, k_\delta)$ can be determined using the PHT using no more than

$$((d-1)k_\delta + 1)\left(1 + \frac{3}{\delta}\right)^d + O\left(\frac{d^{d+1}k_\delta^{2d}}{\delta^{2d(d-1)}}\right)$$

$(1 \cdot 4 + 1)(1 + \frac{3}{1})^2 + O(\quad)$

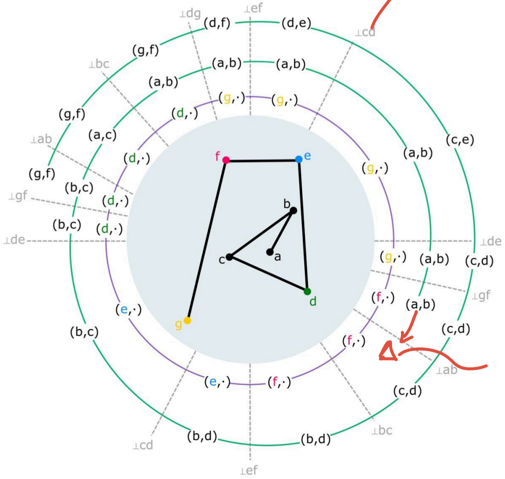
# of directions to check

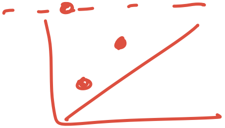
From: Curry, Mukherjee, Turner. How many directions determine a shape and other sufficiency results for two topological transforms. 2022.

Section 4

A bigger example

Monodromy Example





diagonal
$\Delta = \{(x,x) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$

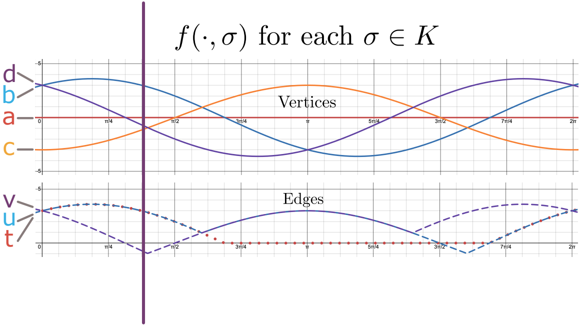
Figure from Arya et al., Decomposing the Persistent Homology Transform of Star Shaped Objects 2024.

Fibered filtration function

US: $B = S^1$

Definition

A **fibered filtration function** is a continuous function $f : K \times B \to \mathbb{R}$. For each $p \in B$, the function $f_p : K \to \mathbb{R}$ defined by $f_p(\sigma) = f(\sigma, p)$ is a filtration function on K.



$p \in B$

$\omega \in S^1$

From: Hickok, Persistence diagram bundles: A multidimensional generalization of vineyards, 2023.

Persistence Bundle

Definition (Hickok 2023)

- Given $f : K \times B \to \mathbb{R}$
- $E = \{(p, z) \mid p \in B, z \in Dgm_q(f_p)\}$
- $\pi : E \to B$, $\pi(p, z) = p$
- (E, B, π) is the **persistence bundle** of f in degree q

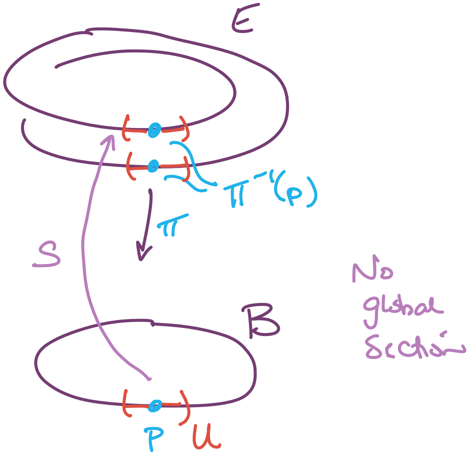
From: Hickok, Persistence diagram bundles: A multidimensional generalization of vineyards, 2023.

Section

Definition

A **local section** of a persistence bundle (E, B, π) over an open set $U \subseteq B$ is a continuous map $s : U \to E$ such that $\pi \circ s = id_U$.

A **(global) section** of a persistence bundle (E, B, π) is a continuous map $s : B \to E$ such that $\pi \circ s = id_B$.



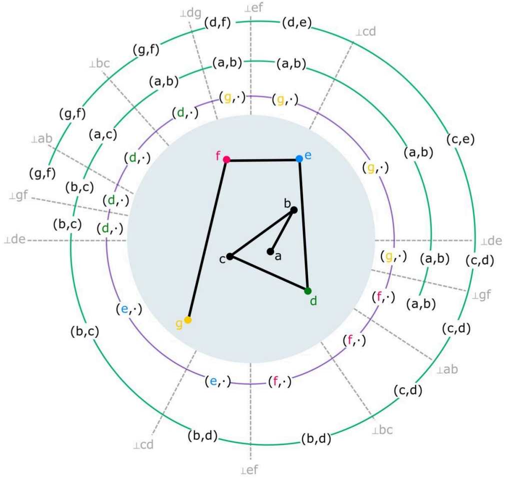
From: Hickok, Persistence diagram bundles: A multidimensional generalization of vineyards, 2023.

Monodromy definition

E
↓
$B = S^1$

Definition

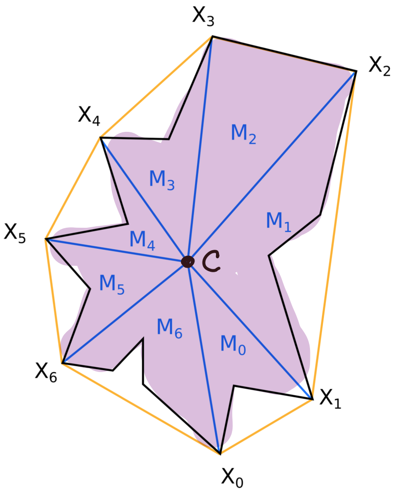
A bundle exhibits non-trivial monodromy if there are no global sections.



Star-Shaped Spaces

Definition

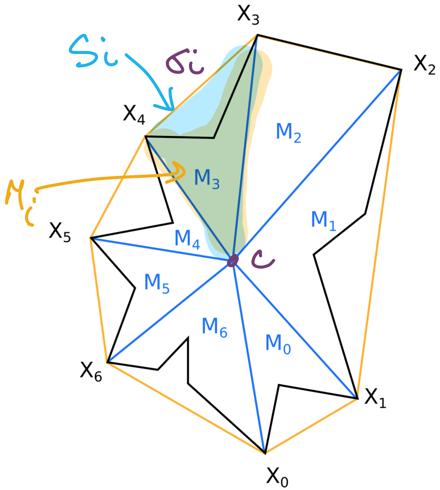
A set $M \subseteq \mathbb{R}^d$ is **star-shaped** if there exists a point $c \in M$ such that for all $x \in M$, the line segment from c to x is contained in M.



From: Arya et al., Decomposing the Persistent Homology Transform of Star Shaped Objects 2024.

Sectors

- Assume the convex hull of M, $C(M)$ is a convex polytope (AKA convex hull of finite set of points).
- Boundary of $C(M)$: $\{\sigma_i\}_{i=1}^n$.
- $S_i = C(\sigma_i \cup \{c\})$
- Sector $M_i = S_i \cap M$.



From: Arya et al., Decomposing the Persistent Homology Transform of Star Shaped Objects 2024.

Star-shaped objects have trivial monodromy

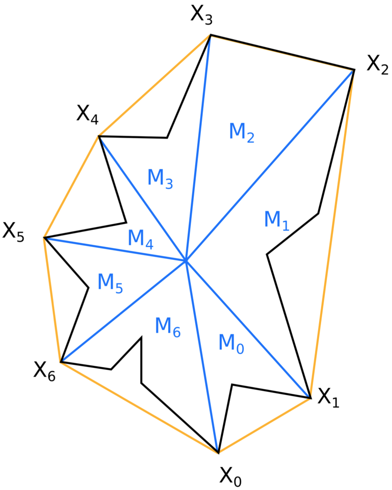
Theorem (Arya et al., 2024)

Let $M \subset \mathbb{R}^2$ be a star-shaped object with sectors $\{M^i\}_{i=0}^m$. The for all $w \in \mathbb{S}^1$,

$$\bigoplus_{i=0}^{m} \widetilde{Pers}_0(M^i, w) \cong \widetilde{Pers}_0(M, w)$$

Theorem (Arya et al., 2024)

Let $M \subset \mathbb{R}^2$ be a star-shaped object. Then the persistence bundle of M in degree 0 has trivial monodromy.



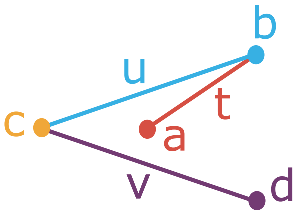
Section 5

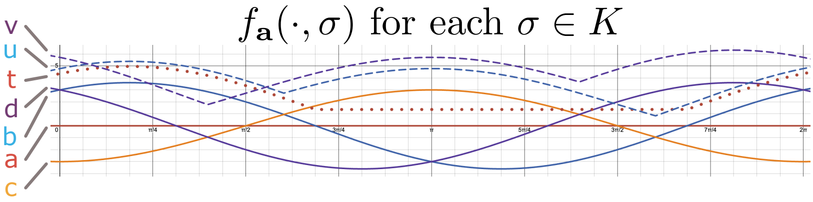
General persistence bundles

Abby's Perturbing a

Definition

- Given $f \times B \to \mathbb{R}$.
- Fix an ordering of the simplices of K, $\{\sigma_1, \ldots, \sigma_n\}$, such that if $\sigma_i < \sigma_j$, then $i < j$.
- Let $A := \{\mathbf{a} \in \mathbb{R}^N \mid a_i \leq a_j, i \leq j\}$
- Define $f_a(\sigma_i, p) = f(\sigma_i, p) + a_i$.



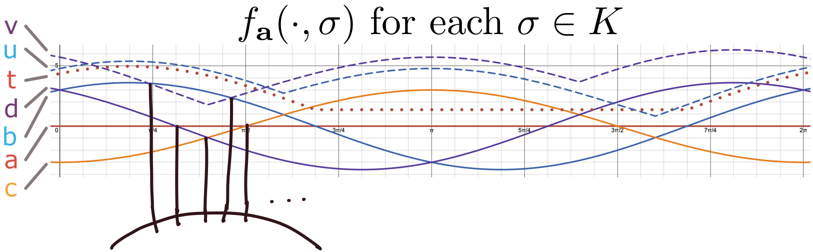


Theorem

for almost every $a \in A$

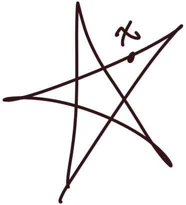
Theorem (Hickok 2023)

For a generic fibered filtration function $f_a \colon K \times B \to \mathbb{R}$ *on a smooth compact manifold* B*, there is a stratification of the* B *with finitely many strata such that the entire bundle is determined by the persistent homology at one point per stratum.*



Next time

-



• z

$A \longmapsto \mathbb{R}$

$f_z(x) = d(x, z)$