Here Comes the Homology

Lecture 5 - CMSE 890

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Goals

Goals for today:

- Ch 2.4: Cycles and Boundaries

Section 1

Linear Algebra Review & Homology

Definition

A field $(k, +, \cdot)$ is a set k with two operations $+$ and $\cdot$ such that for any $a, b, c \in k$:

- Closure: $a+b \in k \qquad a \cdot b \in k$
- Commutativity:
  - $a+b = b+a$
  - $a \cdot b = b \cdot a$
- Associativity:
  - $(a+b)+c = a+(b+c)$
  - $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- Identity:
  - $0_k \Rightarrow a + 0_k = a$
  - $1_k \Rightarrow a \cdot 1_k = a$
- Inverses:
  - $a \quad \exists \; -a \quad$ st. $a+(-a) = 0_k$
  - $a \quad \exists \; \frac{1}{a} \quad$ st $a \cdot (\frac{1}{a}) = 1_k$
- Distributivity:

  $a(b+c) = ab + ac$

Examples:

$(\mathbb{R}, +, \cdot)$

Non ex: $(\mathbb{Z}, +, \cdot)$

$(\mathbb{Z}_2, +, \cdot) = \{0, 1\}$

$0+0=0$
$0+1=1$
$1+1=0$
$0 \cdot 0 = 0$
$0 \cdot 1 = 0$
$1 \cdot 1 = 1$

$\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$

$p = 7 \qquad 5 + 6 \bmod 7 = 4$

Vector space

Definition

A vector space over a field k is a set V with vector addition and scalar multiplication such that $v + w$ $a \cdot v$

- Associative $+$: $(v+w)+x = v+(w+x)$
- Commutative $+$: $v+w = w+v$
- Identity
  - $+$ $0_V \in V$ $w + 0_V = w$
  - $\cdot$ $1_k \in k$ $1_k \cdot v = v$
- Inverses:
  - $+$ Given $v \in V$, $\exists\, -v$ st $v+(-v) = 0_V$

- Scalar vs. field mult:
- Distributivity:
  - $(a+b)v = av + bv$
  - $a(v+w) = av + aw$

Examples:

$V = \mathbb{R}^2$

$k = \mathbb{R}$

$(0+1)(v) = 0 \cdot v + 1v$

$1 \cdot v$

$v = 0 \cdot v + 1 v$

Basis

Definition

A basis for a vector space V is a collection of vectors $\{b_\alpha\}_{\alpha \in A}$ such that

- they are linearly independent and,

  for any $t_\alpha \in k$ $\quad \sum_{\alpha \in A} t_\alpha b_\alpha = 0 \implies t_\alpha = 0 \ \forall \alpha$

- they span V.

Given any $v \in V$, $\exists \{t_\alpha\}_\alpha$ st $\sum t_\alpha b_\alpha = v$

Goal

Build a vector space from a simplicial complex! (Next time)

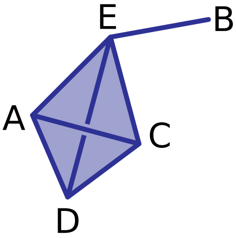
p-Chains

Let K be a simplicial complex and fix a dimension p.
A p-chain is a finite formal sum of p-simplices in K, written

$$\alpha = \sum_{i} a_i \sigma_i$$

p-simplex

elements of $k = \mathbb{Z}_2$



Example 2-chain

$\alpha = AC + CE + BE + 0\,AD + 0 \cdot CD + 0 \cdot DE + 0 \cdot AE$

$\beta = AD + AC + CD$

Addition of chains

p-chains are added component-wise: if $\alpha = \sum a_i \sigma_i$ and $\beta = \sum b_i \sigma_i$, then

$$\alpha + \beta = \sum (a_i + b_i)\sigma_i$$

$$\left.\begin{aligned} \alpha &= AC + CE + BE \\ \beta &= AD + AC + CD \end{aligned}\right\} \begin{aligned} \alpha + \beta &= 1 \cdot AD + \overset{0}{\cancel{(1+1)}}AC + 1 \cdot CE + 1 \cdot BE + 1 \cdot CD \\ &= AD + CE + BE + CD \end{aligned}$$

Chain group

The collection of p-chains with addition is called the p^{th}-chain group (vector space), $C_p(K)$.

Some checks

- Associative $+$: $(\alpha+\beta)+\gamma = \sum(a_i+b_i)\sigma_i + \sum c_i\sigma_i = \sum(a_i+b_i+c_i)\sigma_i = \alpha+(\beta+\gamma)$, with $\alpha = \sum a_i\sigma_i$, $\beta = \sum b_i\sigma_i$, $\gamma = \sum c_i\sigma_i$
- Commutative $+$:
- Zero element: $\sum 0\cdot\sigma_i = 0$
- Inverses: Given $\alpha = \sum a_i\sigma_i$ $\quad -\alpha = \sum(-a_i)\sigma_i$

$\alpha = AC + CE + BE$

In $\mathbb{Z}_2$! $-\alpha = \alpha$

In $\mathbb{R}$ $-\alpha = -AC - CE - BE$

Linear Transformations

A linear transformation between two vector spaces V and W is a map $T : V \to W$ such that the following hold:

- $T(v + w) = T(v) + T(w)$
- $T(a \cdot v) = a \cdot T(v)$

$$v = \sum a_i b_i$$

Matrix representation: picked a basis for V and W

$\{b_1, \ldots, b_n\}$ $\{c_i\}$

$$\overset{T}{\begin{pmatrix} | & | & & | \\ Tb_1 & Tb_2 & \cdots & Tb_n \\ | & | & & | \end{pmatrix}} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

Boundary maps[1]

$$\begin{array}{ccc} C_3(K) & \longrightarrow & C_2(K) \\ \| & & \\ 0 & \longmapsto & 0 \end{array}$$

$$\begin{array}{rccc} \partial_p : & C_p(K) & \to & C_{p-1}(K) \\ & \sigma = \underbrace{[v_0, \cdots, v_p]} & \mapsto & \sum_{j=0}^{p} \underbrace{[v_0, \cdots, \widehat{v_j}, \cdots v_p]}_{\text{remove } \hat{v}_j \text{ from the list}} \end{array}$$

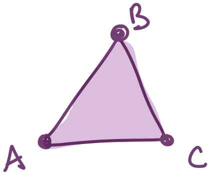
basis $\{AB, AC, BC\}$

$$C_1(K) \longrightarrow C_0(K)$$

$$AB \longmapsto [\hat{A}, B] + [A, \hat{B}] = B + A$$

$$AC \longmapsto [\hat{A}, C] + [A, \hat{C}] = C + A$$

$$BC \longmapsto [\hat{B}, C] + [B, \hat{C}] = C + B$$



$$C_2(K) \longrightarrow C_1(K)$$

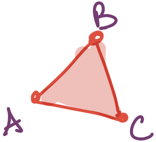
$$[A, B, C] \longmapsto [\hat{A}, B, C] + [A, \hat{B}, C] + [A, B, \hat{C}]$$

$$= BC + AC + AB$$

[1]Warning: We are assuming $\mathbb{Z}_2$ coefficients from now on!

Checking linearity

$$\partial_p(\alpha + \beta) = \partial_p(\alpha) + \partial_p(\beta); \qquad \alpha = \sum a_i \sigma_i, \qquad \beta = \sum b_i, \sigma_i$$



$$C_1(K) \xrightarrow{\partial_1} C_0(K)$$

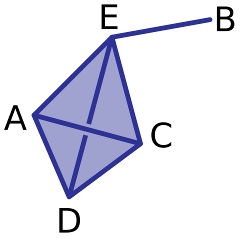
$$\alpha = AB + BC \longmapsto A + B + B + C = A + C$$

$$\beta = BC + AC \longmapsto B + C + A + C = A + B$$

$$\alpha + \beta = AB + 2BC + AC$$

$$= AB + AC \longmapsto A + B + A + C = B + C$$

Try it:



- $\partial_1([a,e]) =$
- $\partial_1([a,e]+[b,e]) =$
- $\partial_1([a,e]+[c,e]+[c,d]+[a,d]) = 0$
- $\partial_2([a,c,e]+[a,c,d]) =$ AE + CE + AD + CD

Ian's homework