Discrete Morse Theory

Lecture 22 - CMSE 890

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This lecture

- DW - Chapter 10.1-10.2
- Additional examples from:
  - Scoville, *Discrete Morse Theory*, 2019.
  - Knudsen, *Morse Theory: Smooth and Discrete*, 2015

Section 1

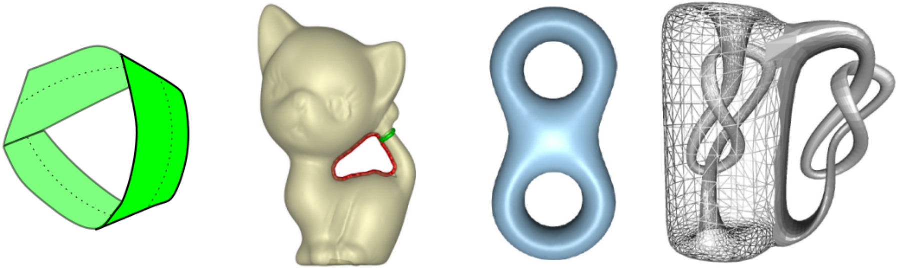
Recall: Manifolds

Manifold definition

Definition

A topological space M is an m-manifold if every point $x \in M$ has a point homeomorphic to the m-ball $\mathbb{B}_o^d$ or the m-hemisphere $\mathbb{H}^d$.

$\mathbb{B}_o^d = \{y \in \mathbb{R}^d \mid \|y\| < 1\}$
$\mathbb{H}^d = \{y \in \mathbb{R}^d \mid d(y, 0) < 1 \text{ and } y_d \geq 0\}$.



(a) (b) (c) (d)

Gradients

Definition

Given a smooth function $f : \mathbb{R}^d \to \mathbb{R}$, the gradient vector field $\nabla f : \mathbb{R}^d \to \mathbb{R}^d$ at x is:

$$\nabla f(x) = \left[\frac{\partial f}{\partial x_1}(x), \cdots, \frac{\partial f}{\partial x_d}(x)\right]$$

Note: This defnition can be extended to more general settings $f : M \to \mathbb{R}$.

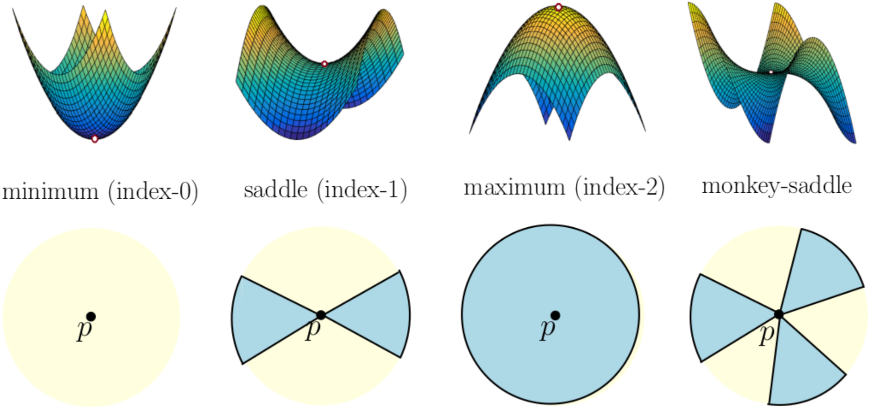
Ex. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = x^2 + y^2$ at $(0, 0)$ and $(1, 0)$

$\nabla f = (2x, 2y) \rightsquigarrow \nabla f(0,0) = (0,0)$

Critical points

- Points in $\mathbb{R}^d$ where $\nabla f(p) = [0, \cdots, 0]$

$f: \mathbb{R}^2 \to \mathbb{R}$



"Nice" critical points

Definition

For a smooth m-manifold M, the Hessian matrix of $f : M \to \mathbb{R}$ is the matrix of second order partial derivatives

$$Hessian(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_m}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_m \partial x_1}(x) & \frac{\partial^2 f}{\partial x_m \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_m \partial x_m}(x) \end{bmatrix},$$

A critical point of f is non-degenerate if the Hessian is non-singular (has non-zero determinant); otherwise it is degenerate.

Examples

- $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 - y^2$.
  - Hessian: $H = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$
  - The critical point at the origin is not degenerate.
- Monkey saddle: $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^3 - 3xy^2$.
  - Hessian: $H = \begin{bmatrix} 6x & -6y \\ -6y & -6x \end{bmatrix} \rightsquigarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
  - The critical point at the origin is degenerate.

Interactive plot: `https://www.desmos.com/3d/cw0km8przc`

Morse lemma

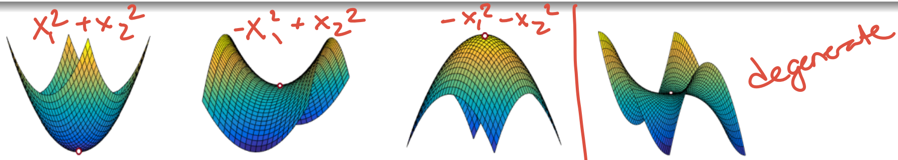
Theorem

Given a smooth function $f : M \to \mathbb{R}$ defined on a smooth m-manifold M with non-degenerate critical point p. There is a local coordinate system in a neighborhood $U(p)$ so that

- $U(p) = (0, \cdots, 0)$
- *Locally any x is of the form*

$$f(x) = f(p) - x_1^2 - \cdots - x_s^2 + x_{s+1}^2 + \cdots + x_m^2.$$

In this case, the integer s is called the index of the critical point p.



minimum (index-0) saddle (index-1) maximum (index-2) monkey-saddle

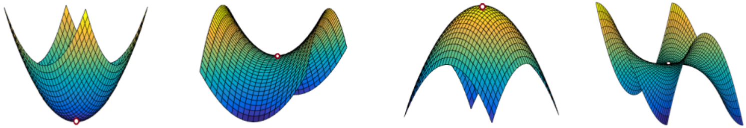
Morse Functions

Definition

A smooth function $f : M \to \mathbb{R}$ defined on a smooth manifold M is a Morse function if

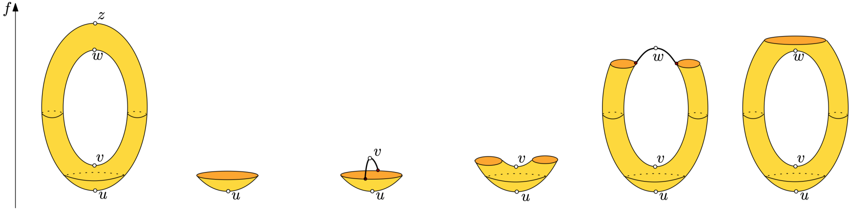
- none of f's critical points are degenerate
- the critical points have distinct function values.

Why do I care? Every function is almost Morse.



minimum (index-0) saddle (index-1) maximum (index-2) monkey-saddle

Sublevelsets



(a) (b) (c) (d) (e) (f)

Crossing critical values changes topology of sublevelsets

Section 2

Discrete Morse Theory

Discrete Morse Function

Definition (Forman, 1995)

A function $f : K \to \mathbb{R}$ is a discrete morse function if for every p-simplex $\sigma_p \in K$,

- $\#\{\sigma_{p-1} \mid \sigma_{p-1} < \sigma_p, f(\sigma_{p-1}) \geq f(\sigma_p)\} \leq 1$
- $\#\{\sigma_{p+1} \mid \sigma_{p+1} > \sigma_p, f(\sigma_{p+1}) \leq f(\sigma_p)\} \leq 1$

Proposition

If a pair (σ_{p-1}, σ_p) satisfies the first condition, there is no pair (σ_p, σ_{p+1}) satisfying the second condition and vice versa.

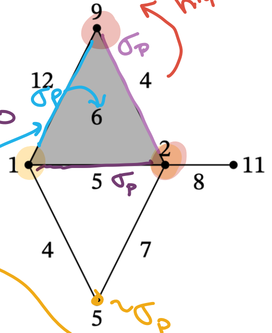


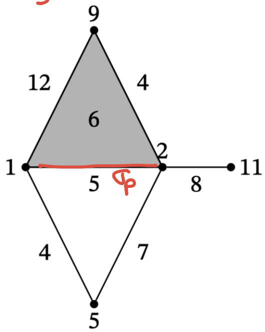
Figure: Scoville 2019

Translation

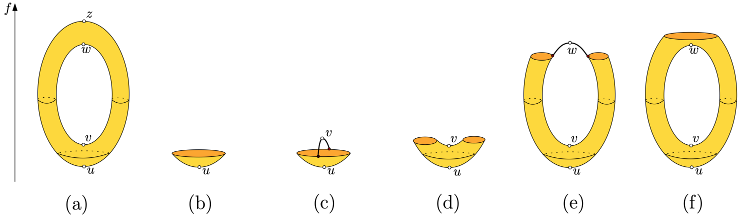
Fix σ_p

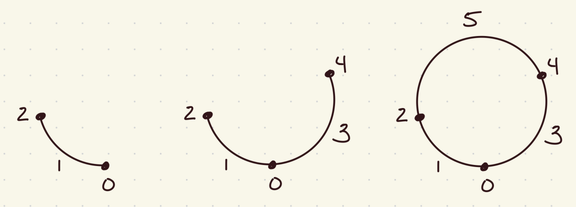
- $\#\{\sigma_{p-1} \mid \sigma_{p-1} < \sigma_p, f(\sigma_{p-1}) \geq f(\sigma_p)\} \leq 1$
- $\#\{\sigma_{p+1} \mid \sigma_{p+1} > \sigma_p, f(\sigma_{p+1}) \leq f(\sigma_p)\} \leq 1$

- Higher dim neighbors have higher values (with ≤ 1 exception)
- Lower dim neighbors have lower values (with ≤ 1 exception)



Intuituon

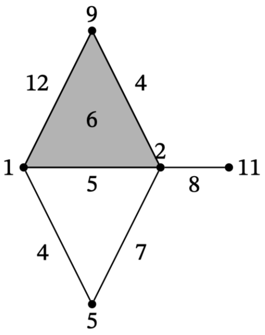




- If $f(u) < f(e) < f(v)$, then adding e and v does not change the homology
- If $f(e) > f(v), f(u)$, then adding e changes the homology

Matching

- $\#\{\sigma_{p-1} \mid \sigma_{p-1} < \sigma_p, f(\sigma_{p-1}) \geq f(\sigma_p)\} \leq 1$
- $\#\{\sigma_{p+1} \mid \sigma_{p+1} > \sigma_p, f(\sigma_{p+1}) \leq f(\sigma_p)\} \leq 1$



Definition

A set of ordered pairs $M = \{(\sigma, \tau)\}$ is a matching in K if the following conditions hold:

- For any $(\sigma, \tau) \in M$, σ is a facet of τ.
- Any simplex in K can appear in at most one pair in M.

$\sigma \leq \tau$

$\dim(\sigma) = \dim(\tau) - 1$

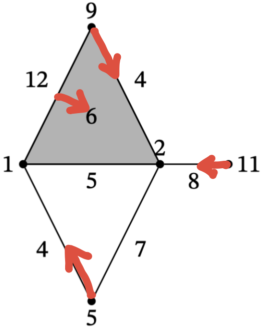
Induced Gradient Vector Field

Definition

Given a discrete Morse function $f : K \to \mathbb{R}$, the induced discrete gradient vector field is the matching

$$V_f = \{(\sigma_p, \tau_{p+1}) \mid \sigma < \tau, f(\sigma) \geq f(\tau)\}.$$

Drawing: We put an arrow from σ to τ for each $(\sigma, \tau) \in M$.

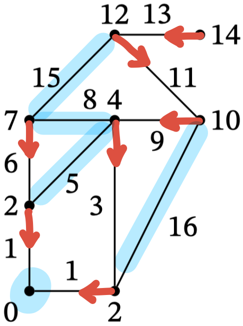


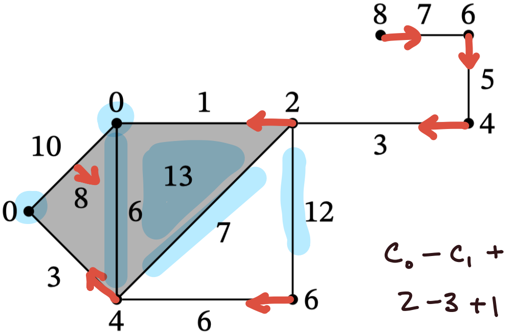
(σ_5, σ_4) (σ_{11}, σ_8)

(σ_{12}, σ_6)

(σ_9, σ_4)

What is the discrete gradient vector field induced by the following discrete Morse functions?





$c_0 - c_1 + c_2$

$2 - 3 + 1 = 0$

$s_0 - s_1$

$1 - 1 = 0$

Figure: Scoville 2019

Discrete Gradient Path

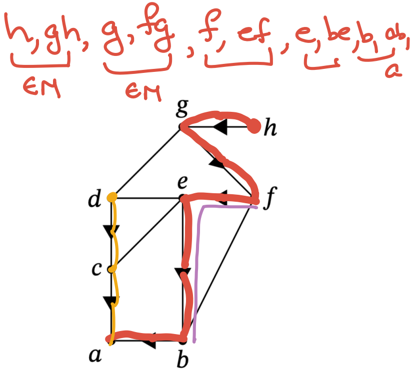
Definition

A V-path (also called a discrete gradient path) of a matching M is a sequence of simplices

$$\sigma_0^{(p)}, \tau_0^{(p+1)}, \sigma_1^{(p)}, \tau_1^{(p+1)}, \cdots, \tau_{r-1}^{(p+1)}, \sigma_r^{(p)}$$

such that for each $i = 0, \cdots, r-1$,

- $(\sigma_i, \tau_i) \in M$
- σ_{i+1} is a facet of τ_i and $\sigma_{i+1} \neq \sigma_i$.



Vocab: σ_{p-1} is a *facet* of σ_p if $\sigma_{p-1} < \sigma_p$ and $\dim(\sigma_{p-1}) = \dim(\sigma_p) - 1$.

Acyclic Matching

Definition

A V-path is

- closed if $\sigma_0 = \sigma_r$
- non-trivial if $r \geq 1$
- maximal if it is not a proper subpath of any other V-path.

A matching M is acyclic if there are no non-trivial closed V-paths.
An acyclic matching is also called a discrete gradient vector field.

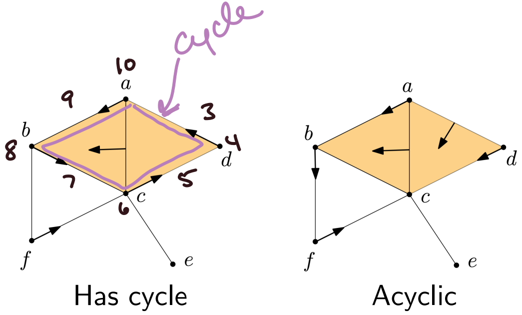
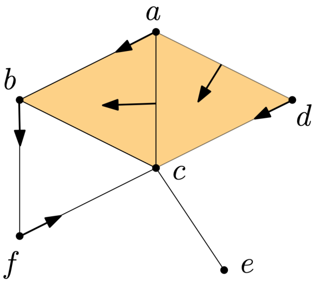


Figure: Dey Wang

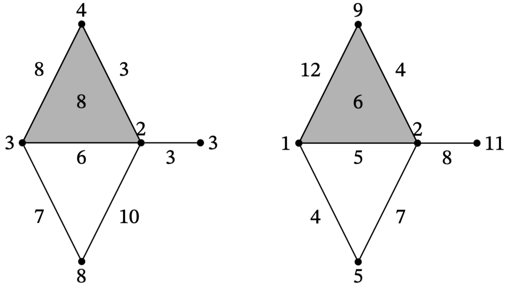
Back and forth

Theorem

A matching is the induced gradient vector field of a discrete Morse function if and only if it is acyclic.



Acyclic example



Non-uniqueness

Forman's Motivation

Identify a simpler complex with the same homology

Definition

A critical simplex σ is a simplex that is not paired with any other simplex in the discrete gradient vector field V.

- Let $M_p \subset C_p(K)$ be the set of critical p-simplices.
- There are maps $\tilde{\partial}_p : M_p \to M_{p-1}$ giving chain complex

Theorem (Forman 1995)

The homology of this chain complex is isomorphic to the homology of K.

Collapses

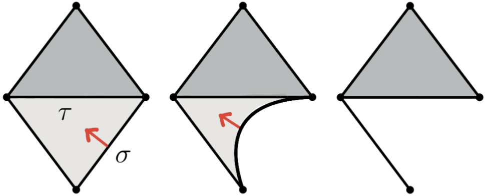
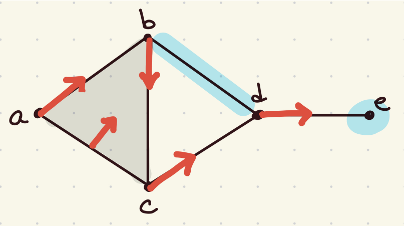


Figure: Fernandez, arXiv2504.15729 2025

Example



$\beta_0 = 1$

$\beta_1 = 1$

$\beta_2 = 0$

Inequalities

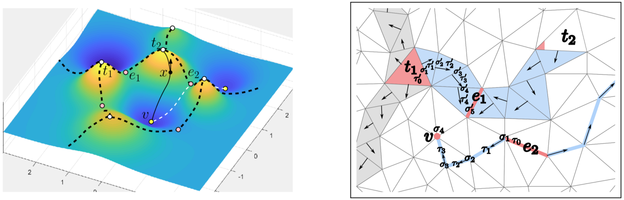
Proposition

Given a Morse function $f : K \to \mathbb{R}$ *with* $c_i =$ *the number of critical* i*-simplices and* $\dim(K) = p$*, then*

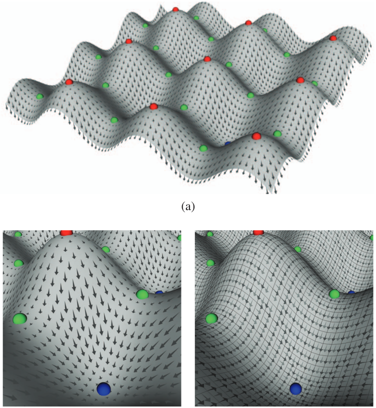
- *Weak Morse inequalities:*
  - $c_i \geq \beta_i$ *for all* $i \geq 0$
  - $c_p - c_{p-1} + \cdots \pm c_0 = \beta_p - \beta_{p-1} + \cdots \pm \beta_0$ $= \pm \chi(K)$
- *Strong Morse inequalities:*
  - $c_k - c_{k-1} + \cdots \pm c_0 \geq \beta_k - \beta_{k-1} + \cdots \pm \beta_0$ *for all* $k \geq 0$

Note: Can derive the weak from the strong.

- Each simplex "flows" to at most one neighbor
- Flow lines go down
- Flow vanishes at critical simplices



(a) (b)

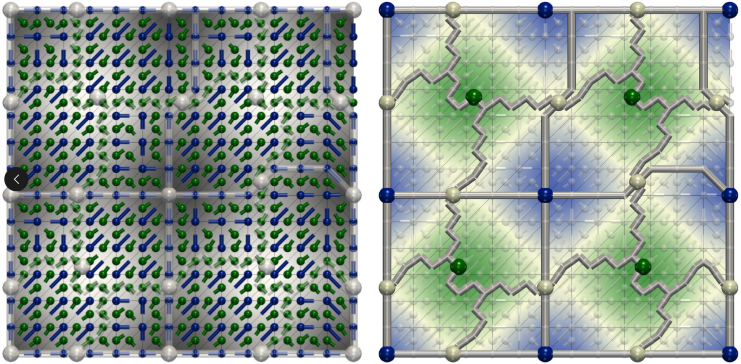


(a)

(b) (c)

Figure: Shivashankar, *Parallel Computation of 2D Morse-Smale Complexes*, 2012

Morse-Smale Complex



Section 3

Uses of Discrete Morse Theory

Simplification of Function Based on Persistence

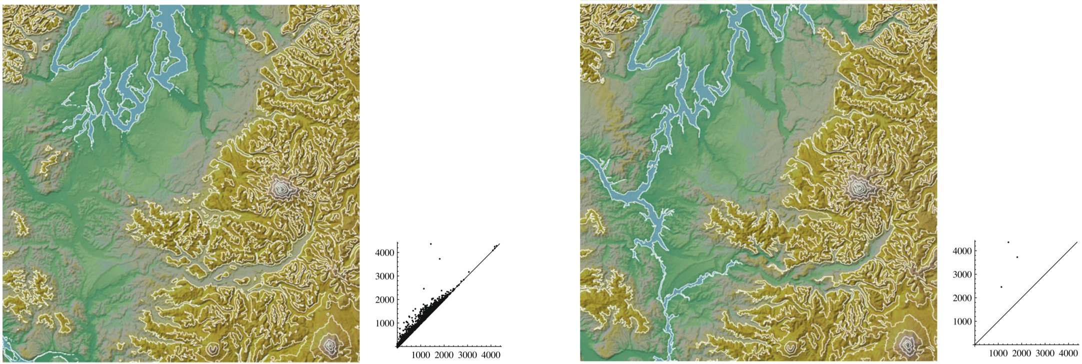
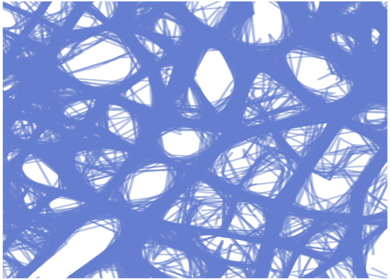
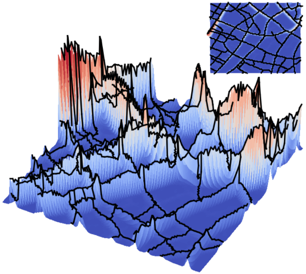


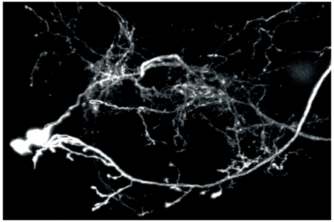
Figure: Bauer et al. 2012

Reconstruction of Road Networks

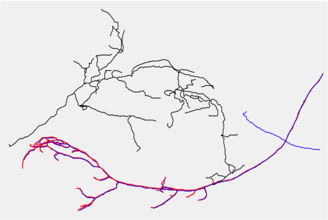




Neuron reconstruction



Input image



Reconstructed neurons

Computation - TTK

https://topology-tool-kit.github.io/