Here Comes the Homology

Are we actually going to get there?
Lecture 6 - CMSE 890

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Goals

Goals for today:

- Homology!

Section 1

More on the boundary map

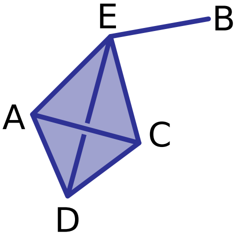
p-Chains

0 - simps = vertices
1 - = edges
2 - = triangles

Let K be a simplicial complex and fix a dimension p.

- A p-chain is a formal sum of p-simplices in K, written
$$\alpha = \sum a_i \sigma_i$$
- p-chains are added component-wise: if $\alpha = \sum a_i \sigma_i$ and $\beta = \sum b_i \sigma_i$, then $\alpha + \beta = \sum (a_i + b_i)\sigma_i$
- The collection of p-chains with addition is called the p^{th}-chain group (vector space), $C_p(K)$.

$C_p(K)$

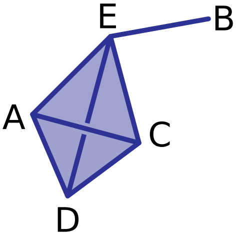


Boundary maps[1]

$$\begin{array}{rccc} \partial_p: & C_p(K) & \to & C_{p-1}(K) \\ & \sigma = [v_0, \cdots, v_p] & \mapsto & \sum_{j=0}^{p} [v_0, \cdots, \widehat{v_j}, \cdots v_p] \end{array}$$

[1] Warning: We are assuming $\mathbb{Z}_2$ coefficients from now on!

Matrix representation



$$\partial_1(K) : C_1(K) \longrightarrow C_0(K)$$

$$\partial_1(K) = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccccc} AC & AD & AE & BE & CD & CE & DE \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Chain complex

$$\cdots \xrightarrow{\partial_{p+2}} C_{p+1}(X) \xrightarrow{\partial_{p+1}} C_p(X) \xrightarrow{\partial_p} C_{p-1}(X) \xrightarrow{\partial_{p-1}} \cdots$$

$\cdots \to C_0(X)$

$$\begin{array}{rccc} \partial_p : & C_p(K) & \to & C_{p-1}(K) \\ & \sigma = [v_0, \cdots, v_p] & \mapsto & \sum_{j=0}^{p} [v_0, \cdots, \widehat{v_j}, \cdots v_p] \end{array}$$

Section 2

Cycles and Boundaries

Important subspaces for a linear transformation

$T: V \longrightarrow W$

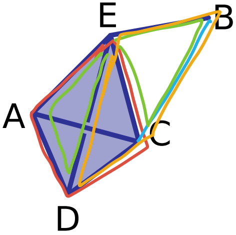
- Image $\mathrm{Im}(T) = \{ w \in W \mid \exists\, v \in V,\ T(v) = w \}$
- Kernel $\mathrm{Ker}(T) = \{ v \in V \mid T(v) = 0 \}$

Cycles

A chain in the kernel of ∂_p is called a p-**cycle**. $C_{p+1}(K) \xrightarrow{\partial_{p+1}} C_p(K) \xrightarrow{\partial_p} C_{p-1}(K)$

$Z_p(K) \subseteq C_p(K)$

$\alpha \longmapsto 0$



$\alpha = \underline{AE} + \underline{CE} + CD + \underline{AD} = 0$

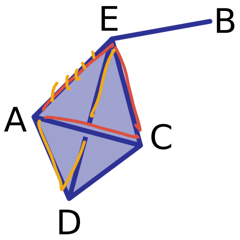
$\alpha \in \mathrm{Ker}(\partial_p)$ call it a cycle

$\beta = \underline{AE} + ED + \underline{AD} + \underline{EC} + BC + BE$

$\alpha + \beta = CD + ED + BC + BE$

The collection of p-cycles forms a subspace $Z_p(K) \subseteq C_p(K)$.

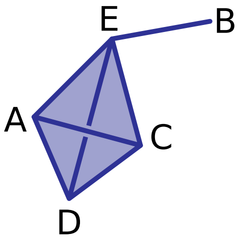
What is a 2-cycle? element of kernel of ∂_2



$$C_2(K) \xrightarrow{\partial_2} C_1(K)$$

$$ACE + ADE + DCE + ACD \longmapsto \cancel{AC} + \cancel{CE} + \cancel{AE} + \cancel{AD} + \cancel{DE} + \cancel{AE} + \cancel{CD} + \cancel{DE} + \cancel{CE} + \cancel{AC} + \cancel{CD} + \cancel{AD} = 0$$

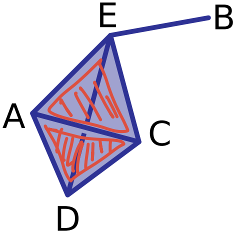
More work space if needed



Boundaries

$B_p(K)$
$\cap|$

A chain in the image of ∂_{p+1} is called a p-**boundary**. $C_{p+1}(K) \xrightarrow{\partial_{p+1}} C_p(K) \xrightarrow{\partial_p} C_{p-1}(K)$



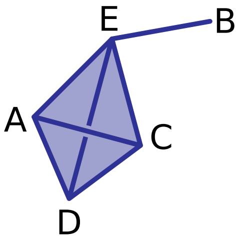
$B_p(K) = \{\alpha \in C_p(K) \mid \exists\, \gamma \in C_{p+1}(K),\ \partial_{p+1}(\gamma) = \alpha\}$

$C_2(K) \longrightarrow C_1(K) \longrightarrow C_0(K)$

$ACD + ACE \longmapsto \underbrace{AE + CE + CD + AD}_{\in B_1(K)} \longmapsto \cancel{A} + \cancel{E} + \cancel{C} + \cancel{E} + \cancel{C} + \cancel{D} + \cancel{A} + \cancel{D} = 0$

The collection of p-boundaries forms a subspace $B_p(K) \subseteq C_p(K)$.

More work space



Nifty trick

Theorem

$\partial_p \partial_{p+1}(\alpha) = 0$ *for every* $(p+1)$*-chain* α.

$$C_{p+1}(K) \xrightarrow{\partial_{p+1}} C_p(K) \xrightarrow{\partial_p} C_{p-1}(K)$$

$$\alpha \longmapsto \underbrace{\partial_{p+1}(\alpha)}_{\text{in } B_p(K)} \underset{\partial}{\longmapsto} \partial_p \partial_{p+1}(\alpha) = 0$$

also in $Z_p(K)$

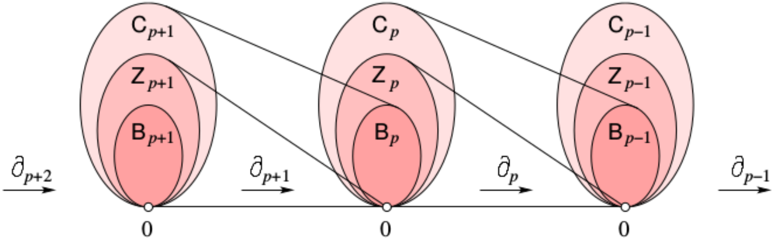
* Any boundary is a cycle, $B_p(K) \subseteq Z_p(K)$

Translation

Every p-boundary is a p-cycle.

$B_p(K) \subseteq Z_p(K) \subseteq C_p(K)$

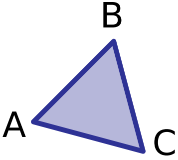
$$C_{p+1}(K) \xrightarrow{\partial_{p+1}} C_p(K) \xrightarrow{\partial_p} C_{p-1}(K)$$



Try it: Cycles and boundaries

$$B_1(K) = \langle AB+BC+AC \rangle$$
$$\rotatebox{-90}{$\subseteq$}$$
$$Z_1(K) = \langle AB+BC+AC \rangle$$
$$\rotatebox{-90}{$\subseteq$}$$

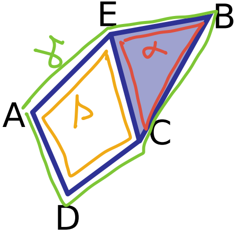
What are the generators of $B_1(K)$? Of $Z_1(K)$?



$$C_2(K) \longrightarrow C_1(K) \longrightarrow C_0(K)$$
$$ABC \longmapsto AB+BC+AC$$

Try it: Cycles and boundaries

What are the generators of $B_1(K)$? Of $Z_1(K)$?



$B_1(K) = \langle BC+CE+BE \rangle$

$\cap$

$Z_1(K) = \langle \overset{\alpha}{BC+CE+BE}, \overset{\beta}{AE+CE+CD+AD} \rangle$

$\cap$

$C_2(K) \longrightarrow C_1(K) \longrightarrow C_0(K)$

$BCE \longmapsto BC+CE+BE$

$\langle \alpha, \gamma \rangle$

<u>All</u> elements of $Z_1(K)$

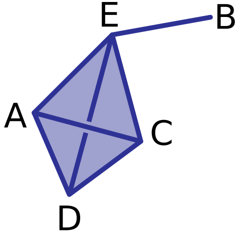
$\{0, \alpha, \beta, \gamma\}$

$\gamma \to \alpha + \beta$

$\beta \to = \gamma + \alpha$

Try it: Cycles and boundaries

What are the generators of $B_1(K)$? Of $Z_1(K)$?



$$C_2(K) \longrightarrow C_1(K) \longrightarrow C_0(K)$$

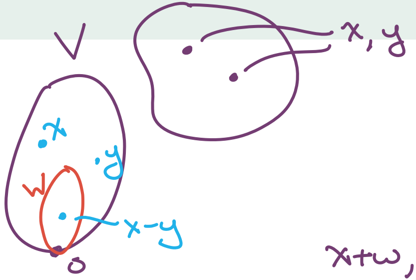
Homework (In case we only get this far)

- DW 2.6.3) Let K be the simplicial complex of a tetrahedron. Write a basis for the chain groups C_1 and C_2; boundary groups B_1 and B_2; and cycle groups Z_1 and Z_2. Write the boundary matrix representing the boundary operator ∂_2 with rows and columns representing bases of C_1 and C_2 respectively.

Section 3

Homology for real now

Let V be a vector space over a field k.
Let $W \subset V$ be a subspace.
Define $\sim$ on V by $x \sim y$ iff $x - y \in W$.
The equivalence class of x is denoted



$$[x] = x + W = \{x + w : w \in W\}.$$

① $x \sim x$
② $x \sim y \Rightarrow y \sim x$
③ $x \sim y,\ y \sim z \Rightarrow x \sim z$

$x+w,\ x+w' \in [x]$

$(x+w) - (x+w') = w - w'$

The quotient space V/W is then defined as $\{[x] \mid x \in V\}$. This is also a vector space with:

- Scalar multiplication: $a \cdot [x] = [a \cdot x]$

  $a \in k = \mathbb{Z}_2$

- Addition: $[x] + [y] = [x+y]$

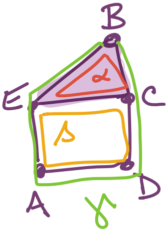
Warning:
many choices of representatives
$[x] = [x']$
$\forall\, x' \in [x]$

Homology

Definition

The p^{th} homology group is the quotient space

$$H_p(K) := Z_p(K)/B_p(K)$$



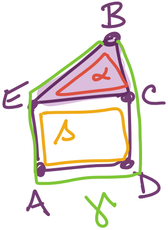
$$C_2(K) \longrightarrow C_1(K) \longrightarrow C_0(K)$$

$$V = \{0, \alpha, \beta, \alpha+\beta\} = Z_1(K) = \langle \alpha, \beta \rangle$$

$$W = \{0, \alpha\} = B_1(K) = \langle \alpha \rangle$$

$$[\beta] = \{\beta + \omega \mid \omega \in B_1(K)\}$$
$$= \{\beta + 0, \beta + \alpha\} = \{\beta, \gamma\}$$

Spare blank page



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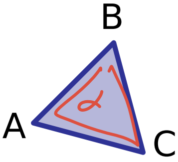
$$[\beta] = \{\beta + \omega \mid \omega \in B_1(K)\}$$
$$= \{\beta + 0, \beta + \alpha\} = \{\beta, \gamma\}$$

$$H_p(K) = \{[0], [\beta]\}$$
$$= \langle [\beta] \rangle$$

$$[\alpha] = \{\alpha + \omega \mid \omega \in B_1(K)\}$$
$$= \{\alpha + 0, \alpha + \alpha\} = \{\alpha, 0\}$$

$$[\gamma] = [\beta] \qquad [0] = [\alpha]$$

Tryit: What is $H_1(K)$?



$$B_1(K) = \langle AB+BC+AC \rangle$$

$$\cap\!\!|$$

$$Z_1(K) = \langle AB+BC+AC \rangle$$

$$\cap\!\!|$$

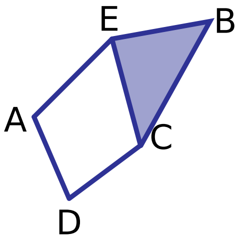
$$C_2(K) \longrightarrow C_1(K) \longrightarrow C_0(K)$$

$$ABC \longmapsto AB+BC+AC$$

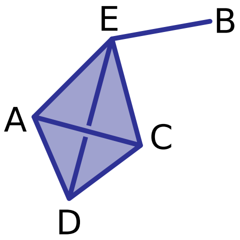
$$[\alpha] = \{0, \alpha\} = [0]$$

$$H_1(K) = 0$$

Tryit: What is $H_1(K)$?



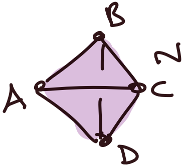
Tryit: What is $H_2(K)$?



Homework

- Almost certainly didn't finish all the examples above....

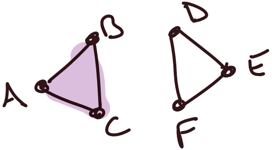
$H_2(K)$



No inside

Atoshi

$H_0(K)$



Zhengrong

$$C_1(K) \longrightarrow C_0(K) \longrightarrow 0$$