Multiparameter Persistence and RIVET

Lecture 23 - CMSE 890

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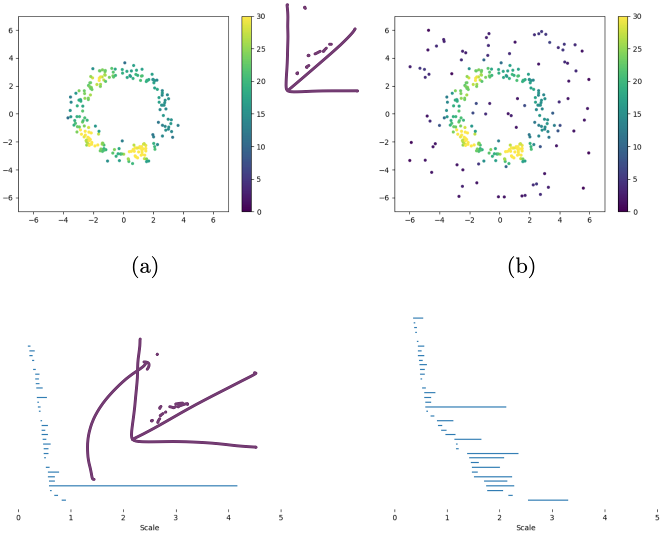
Goals for today

- Multiparamter persistence survey
  - Botnan, M. B., & Lesnick, M. (2022, January 1). An introduction to multiparameter persistence. arXiv:2203.14289
- RIVET
  - `https://rivet.readthedocs.io`
- Crocker plots:
  - Topaz, Ziegelmeier, & Halverson, Topological data analysis of biological aggregation models. PLoS ONE 2015. 10.1371/journal.pone.0126383

Section 1

Mathematical prelimiaries

The problem



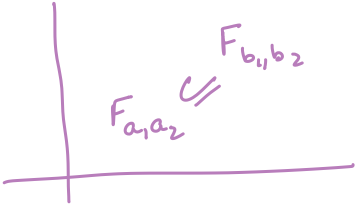
- Colored by local density
- Persistence is not stable to ambient noise

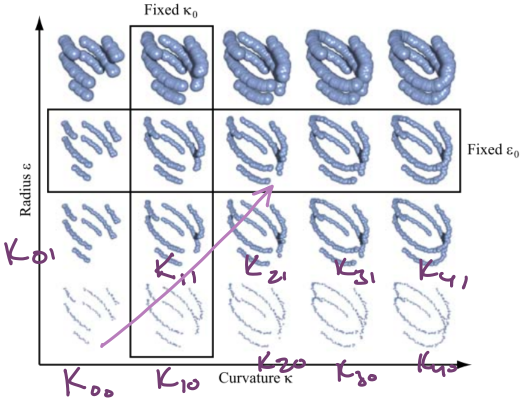
Figure: Botnan and Lesnick 2022 survey

Bifiltration

$K_{a_1} \subseteq K_{a_2} \subseteq K_{a_3} \subseteq \cdots$

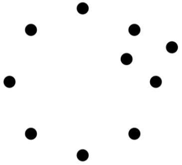
A bifiltration F is a collection of finite simplicial complexes indexed by $\mathbb{Z}^2$ such that $F_{(a_1,a_2)} \subseteq F_{(b_1,b_2)}$ whenever $a_1 \leq b_1$ and $a_2 \leq b_2$.



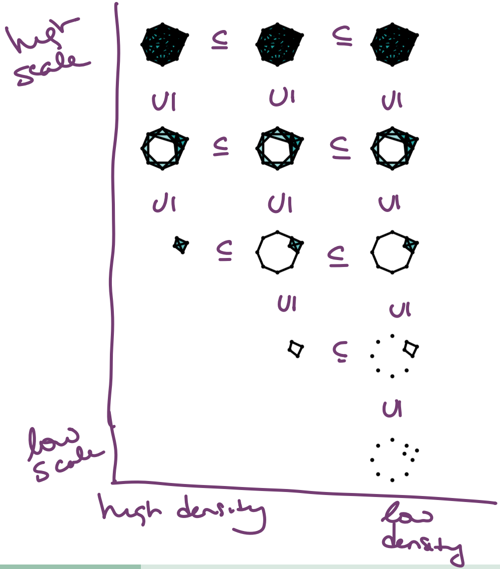


Images: Rivet documentation (left); *The Theory of Multidimensional Persistence* Carlsson & Zomorodian 2009 (right)

Bfiltration from function



- x-axis: filter by density f_x
- y-axis: filter by scale f_y ⇝ Rips
- $K_{a,b} = \{\sigma \in K \mid f_x(\sigma) \leq a, f_y(\sigma) \leq b\}$



Function options implemented in RIVET

- **Ball density function**: (r fixed parameter; C normalization constant so that $\sum \gamma(x) = 1$)

$$\gamma(x) = C \cdot (\# \text{ points in } P \text{ within distance } r \text{ of } x)$$

- **Gaussian density function** ($\sigma > 0$ parameter, C normalization constant)

$$\gamma(x) = C \sum_{y \in P} \exp\left(\frac{-d(x,y)^2}{2\sigma} \right)$$

- **Eccentricity function** ($q \in [1, \infty)$)

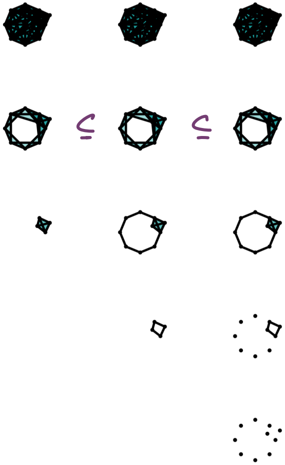
$$\gamma(x) = \left(\frac{\sum_{y \in P} d(x,y)^q}{|P|} \right)^{\frac{1}{q}}$$

Bipersistence module

$\mapsto H_p(K_{(a_1,a_2)})$

M is a collection of vector spaces $\{M_a\}_{a \in \mathbb{Z}^2}$ with maps $\{M_{a,b} : M_a \to M_b\}_{a \leq b}$

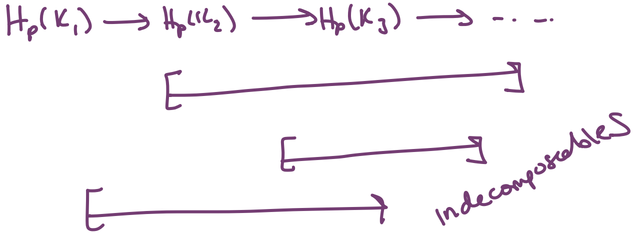
$$\begin{array}{ccccccc}
M_{0,2} & \longrightarrow & M_{1,2} & \longrightarrow & M_{2,2} & \longrightarrow & M_{3,2} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
M_{0,1} & \longrightarrow & M_{1,1} & \longrightarrow & M_{2,1} & \longrightarrow & M_{3,1} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
M_{0,0} & \longrightarrow & M_{1,0} & \longrightarrow & M_{2,0} & \longrightarrow & M_{3,0}
\end{array}$$



No free lunch

Theorem (Carlsson & Zomorodian 2009 (paraphrased))

The algebraic classification of indecomposables of multiparameter persistence modules contain both discrete and continuous portions, meaning that no persistence-diagram-like representation is available.



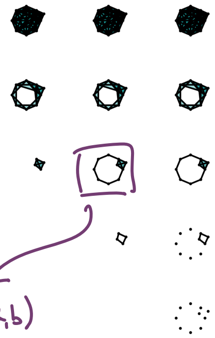
The Theory of Multidimensional Persistence. Carlsson & Zomorodian 2009

Invariants of a bipersistence module

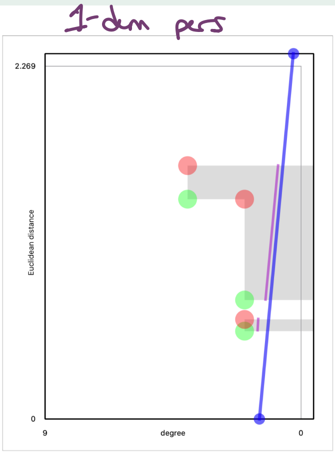
The Hilbert function

The Hilbert function $Hilb^M : \mathbb{R}^2 \to \mathbb{N}$ counts the dimension of the vector space at each location, $(a,b) \mapsto \dim(M_{a,b})$.

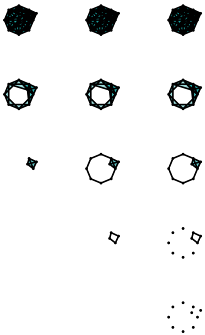
"Changing Betti of $K_{a,b}$"

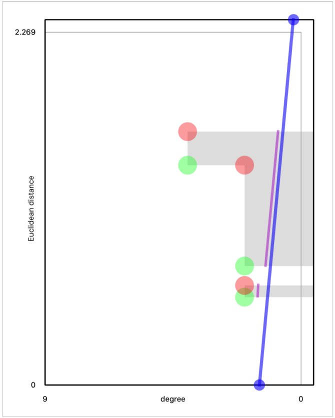


$Hilb^M(a,b) = \dim(H_1(K_{a,b})) = 1$



- The bigraded Betti numbers ζ_i^M. These are functions $\mathbb{R}^2 \to \mathbb{N}$ that, respectively, count the number of births, deaths, and "relations amongst deaths" at each bigrade.
- *Warning: Not the same as the Betti numbers of a space!*
- Related to minimal free resolution of M
- ζ_0: Red dots; ζ_1: Green dots; ζ_2: Yellow dots

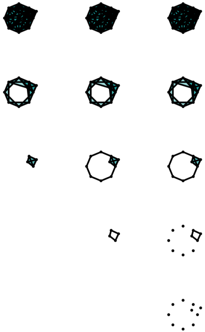


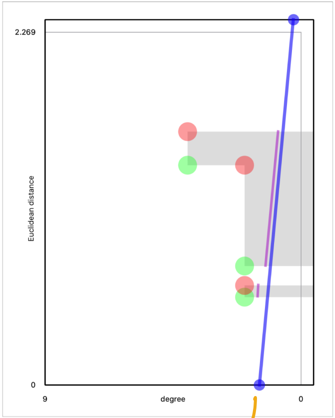


Invariants of a bipersistence module

Fibered Barcode

- Restrict to a line L with positive slope
- Draw the barcode/persistence diagram for that line





1D Pers module → Pers dgm

Example: HIV Data
Option 1

- 0-dimensional persistence
- Metric for Rips: Hamming distance

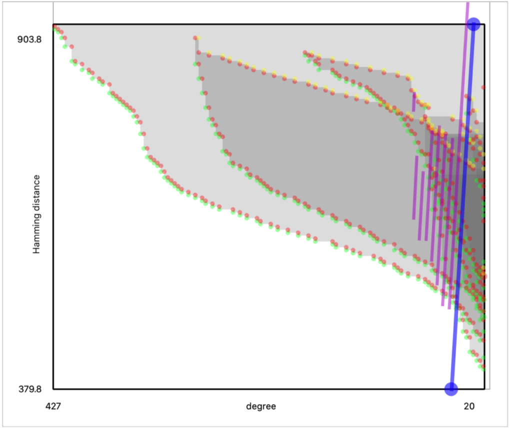


Figure: Botnan and Lesnick 2022 survey; Data: Chan et al. Topology of viral evolution. PNAS 2013.

Example: HIV Data

Option 2

- 0-dimensional persistence
- Metric for Rips: Hamming distance

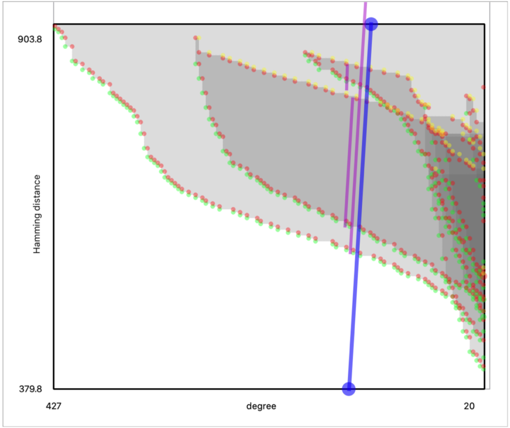


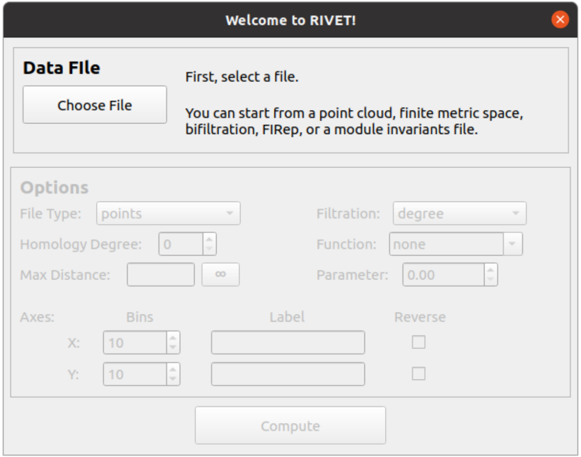
Figure: Botnan and Lesnick 2022 survey; Data: Chan et al. Topology of viral evolution. PNAS 2013.

Section 2

Using RIVET

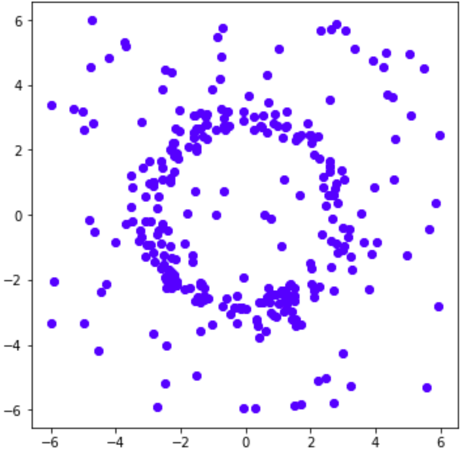
Running the GUI

Run `rivet_GUI`

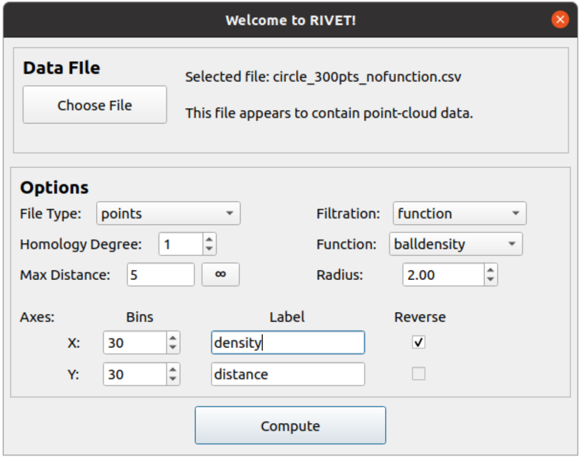


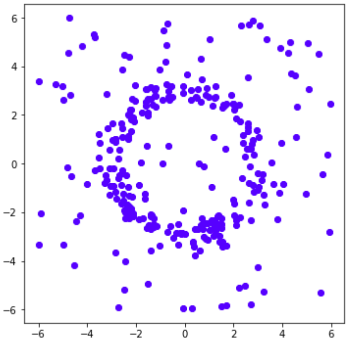
Test data

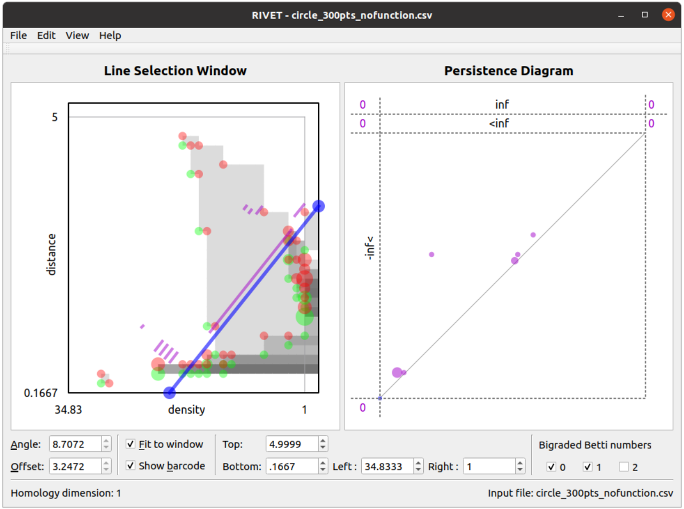
Example included in install: `data/Test_Point_Clouds/circle_300pts_nofunction.csv`



Inputs for this example



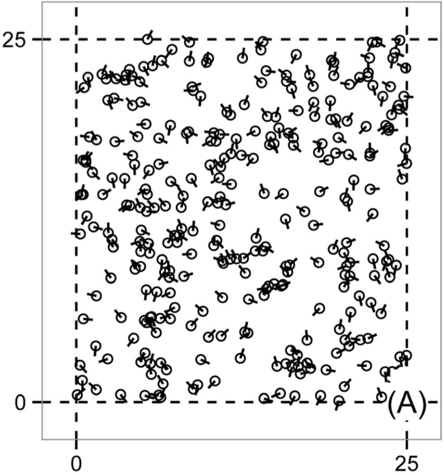

6
4
2
0
−2
−4
−6
−6
−4
−2
0
2
4
6


RIVET - circle_300pts_nofunction.csv
File Edit View Help
Line Selection Window
Persistence Diagram
5
0.1667
distance
34.83
density
1
0
inf
0
0
<inf
0
-inf<
0
Angle: 8.7072
Offset: 3.2472
Fit to window
Show barcode
Top: 4.9999
Bottom: .1667
Left : 34.8333
Right : 1
Bigraded Betti numbers
0
1
2
Homology dimension: 1
Input file: circle_300pts_nofunction.csv

Section 3

Crocker Plots

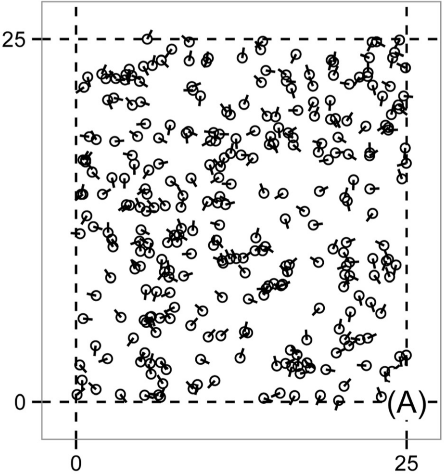
Data: Flocking



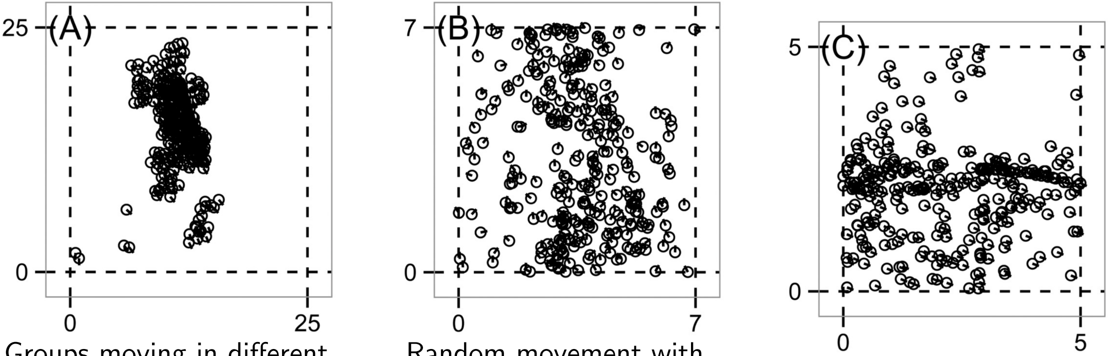
Left: Wikipeda; Right: Topaz et al 2015.

Vicsek Model

- Discrete time, continuous space
- Motion of interacting point particles in a square with periodic boundary conditions
- Updates:
  - θ: Angle of motion updated to average angle of neighbors + noise
  - Position updated by moving in direction of θ with fixed speed
  - Parameters: number of particles, density, noise level, interaction radius, speed



Examples



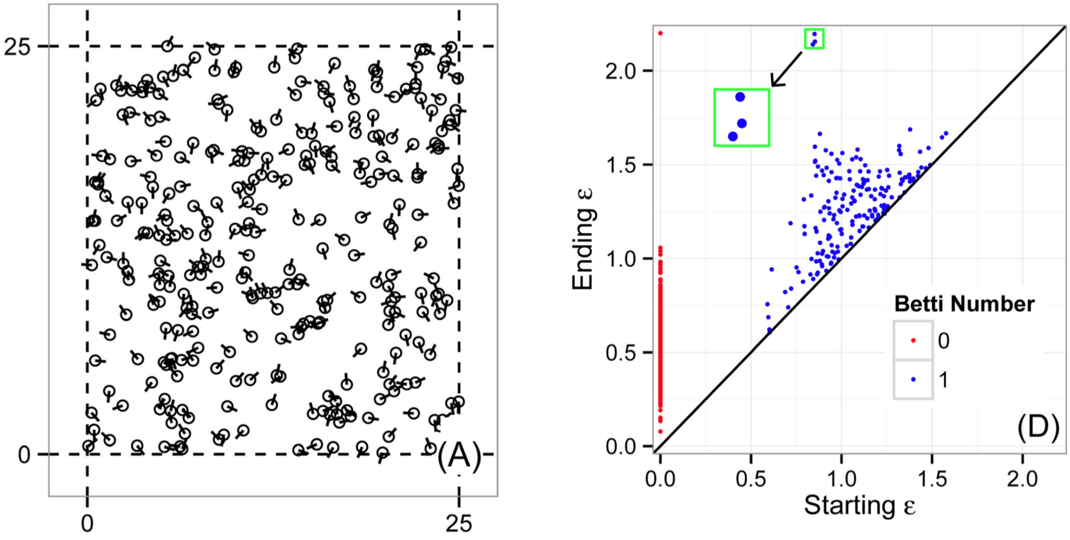
Groups moving in different directions
$\ell = 25$, $\eta = 0.1$

Random movement with some correlation
$\ell = 7$, $\eta = 2$

Highly polarized movement
$\ell = 5$, $\eta = 0.1$

Boxsize ℓ, Noise η

Persistence at each time point



Dynamic Metric Space

Definition 7.1 ([58]) A *dynamic metric space* is a pair $\gamma_X = (X, d_X(\cdot))$ where X is a nonempty finite set and $d_X : \mathbf{R} \times X \times X \to \mathbf{R}_+$ satisfies:

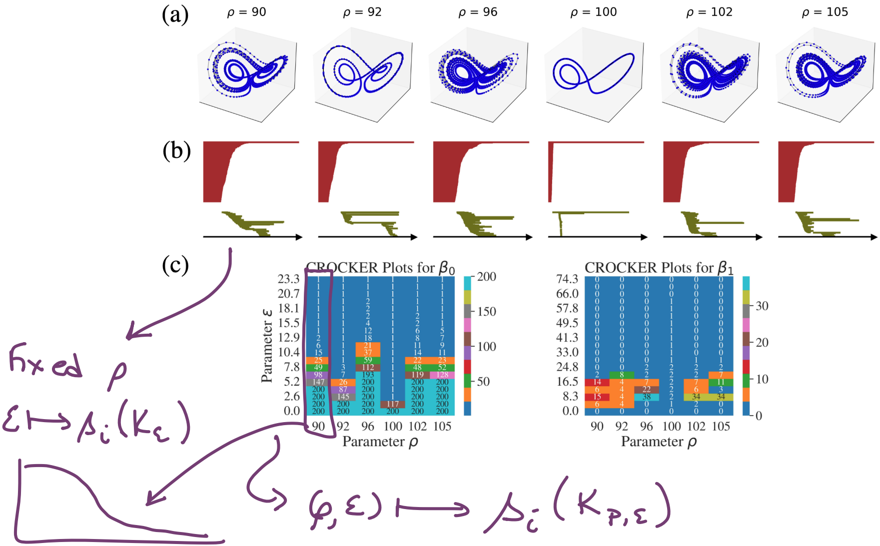
(i) For every $t \in \mathbf{R}$, $\gamma_X(t) = (X, d_X(t))$ is a pseudo-metric space.

(ii) For any $x, x' \in X$ with $x \neq x'$ the function $d_X(\cdot)(x, x') : \mathbf{R} \to \mathbf{R}_+$ is not identically zero.

(iii) For fixed $x, x' \in X$, $d_X(\cdot)(x, x') : \mathbf{R} \to \mathbf{R}_+$ is continuous.

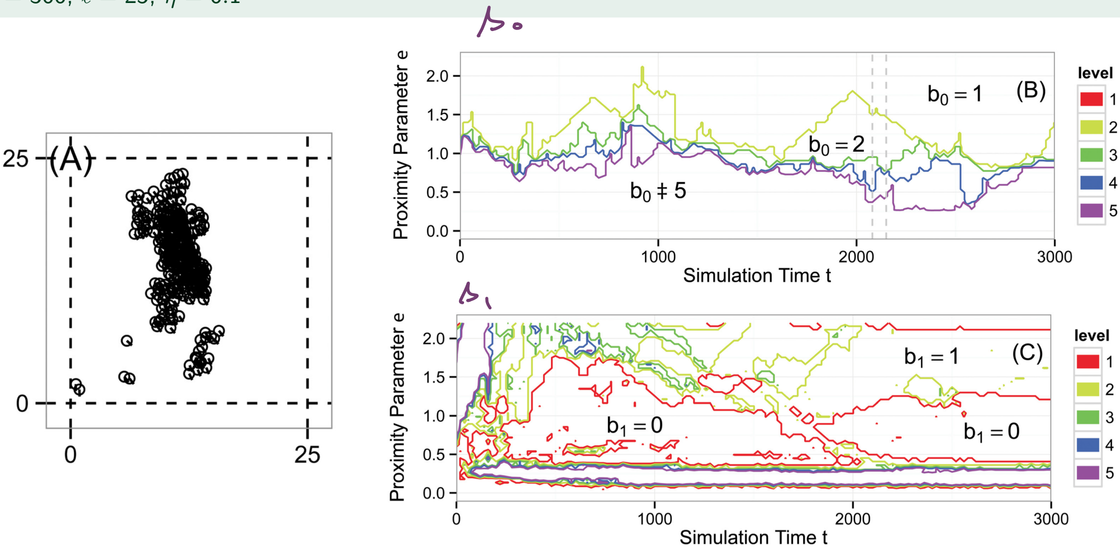
Def: Kim & Mémoli 2024

Crocker Plot



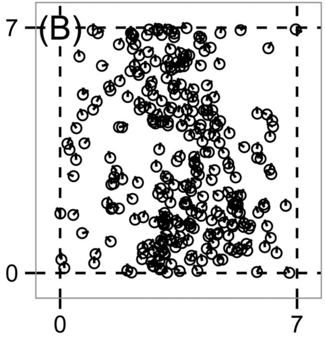
Crocker Plots

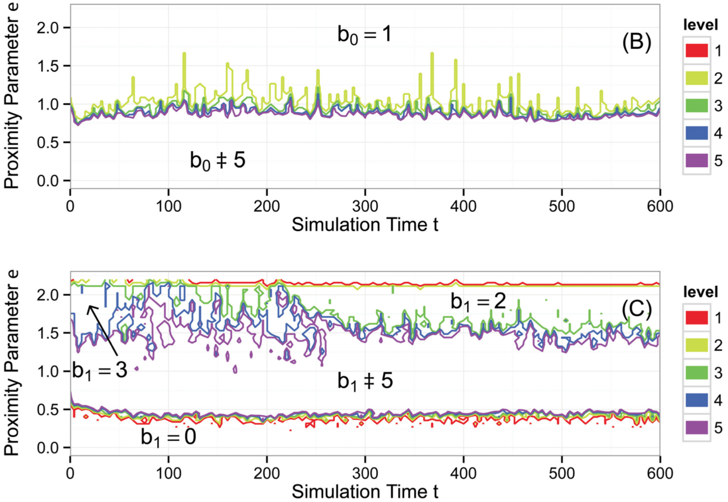
$N = 300$, $\ell = 25$, $\eta = 0.1$



Crocker Plots

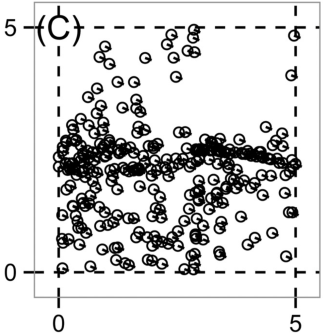
$N = 300$, $\ell = 7$, $\eta = 2$

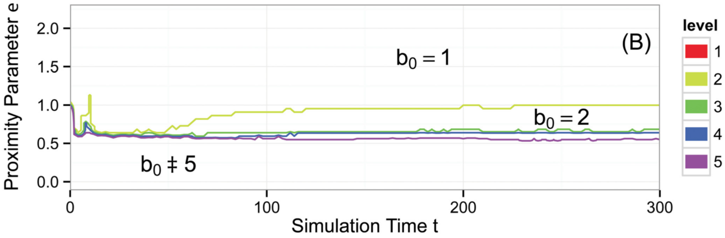


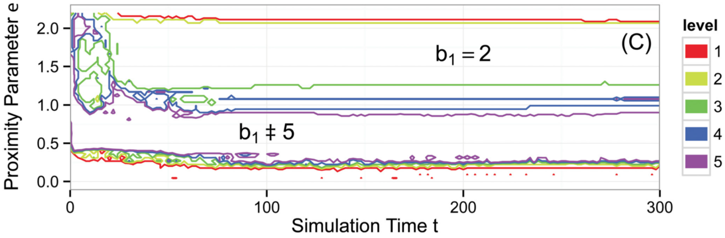


Crocker Plots

$N = 300$, $\ell = 5$, $\eta = 0.1$







Next time

- Time series analysis