Topological Signal Processing

Lecture 24 - CMSE 890

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Dept of Computational Mathematics, Science & Engineering

Thurs, Dec 4, 2025

This lecture

- Time series analysis with persistence

Input: 1D Time series

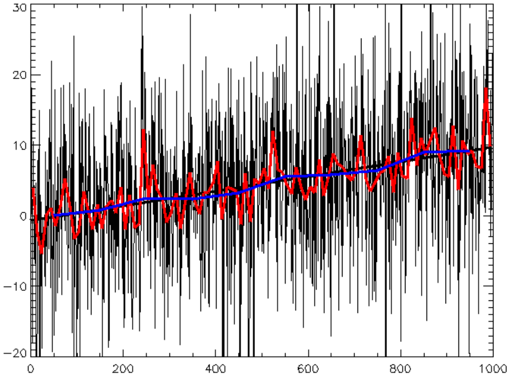
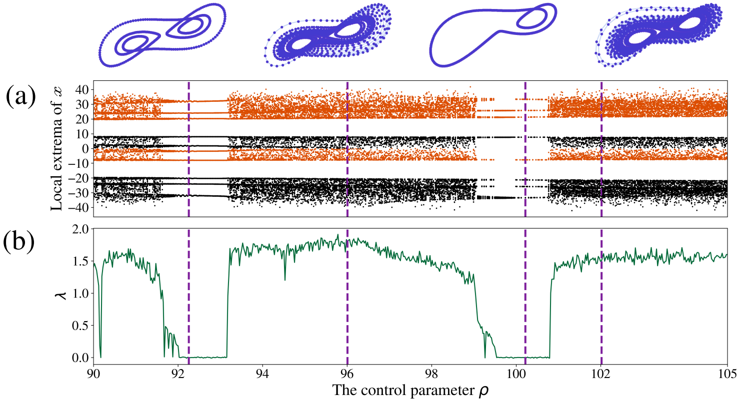


Image: Wikipedia

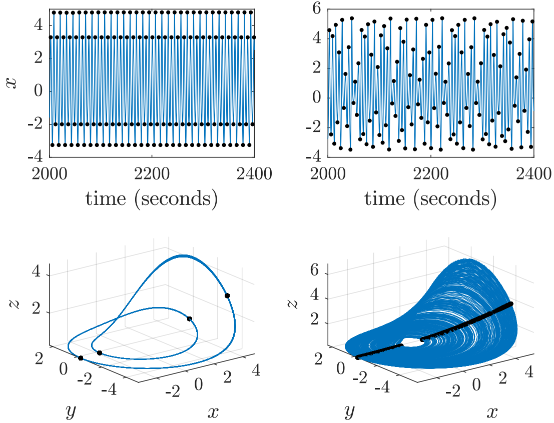
- $f : \mathbb{R} \to \mathbb{R}$
- Discrete samples $\{x_t\}_{t=1}^N$

Lorenz system



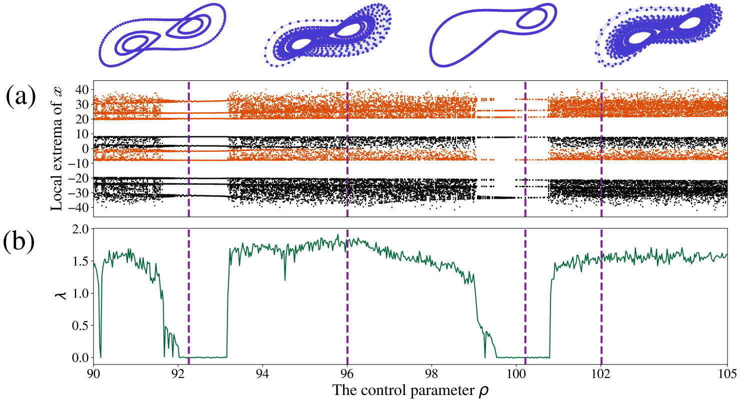
$$\dot{x} = \sigma(y - x),$$
$$\dot{y} = x(\rho - z) - y,$$
$$\dot{z} = xy - \beta z,$$

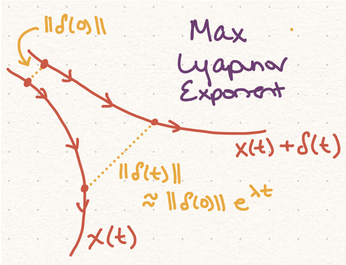
Rossler system



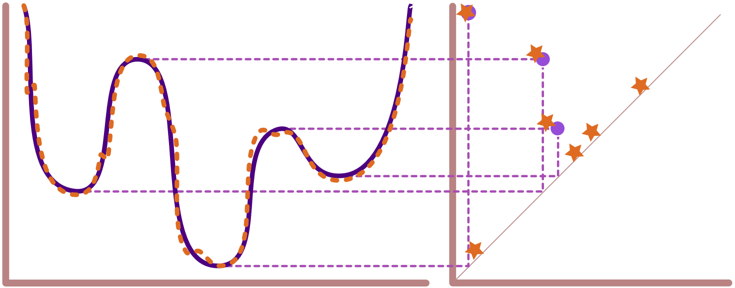
$$\dot{x} = -y - z,$$
$$\dot{y} = x + ay,$$
$$\dot{z} = b + z(x - c),$$

Maximum Lyapunov exponent

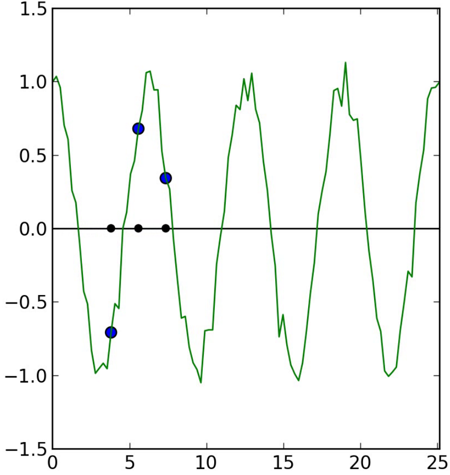


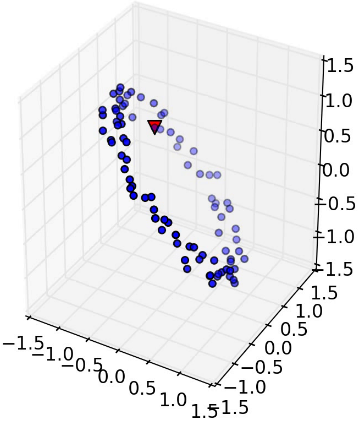


Option 0: Sublevelset persistence of time series



Takens' embedding





$$\Phi_{d,\tau} : [x(t_1), x(t_2), \cdots, x(t_k)] \mapsto \{(x(t)), x(t+\tau), \cdots, x(t+(d-1)\tau)\}$$

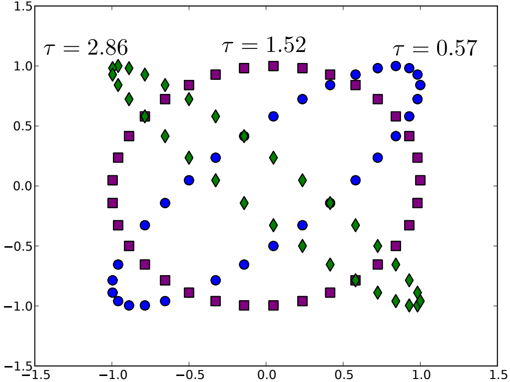
Takens Theorem

- Attractor A
- Given time series data from scalar measurements $s : A \to \mathbb{R}$
- Choose m, τ
- The delay reconstruction is $\{(s_{n-(m-1)\tau}, \ldots, s_{n-\tau}, s_n)\}_n \subseteq \mathbb{R}^m$

Theorem (Takens 1981; Sauer, Yorke, Casdagli 1991)

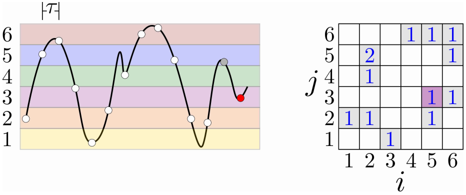
For generic smooth measurement functions s, the map from the attractor A to the delay reconstruction in $\mathbb{R}^m$ is an embedding when $m > 2\mathrm{dim}(A)$ where $\mathrm{dim}(A)$ is the box counting dimension.

Example with different delays



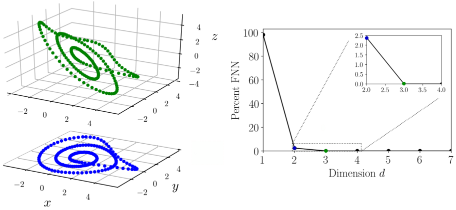
Mutual Information

$$I_\epsilon(\tau) = \sum_{i,j} p_{ij}(\tau) \ln p_{ij}(\tau) - 2 \sum_i p_i \ln p_i,$$



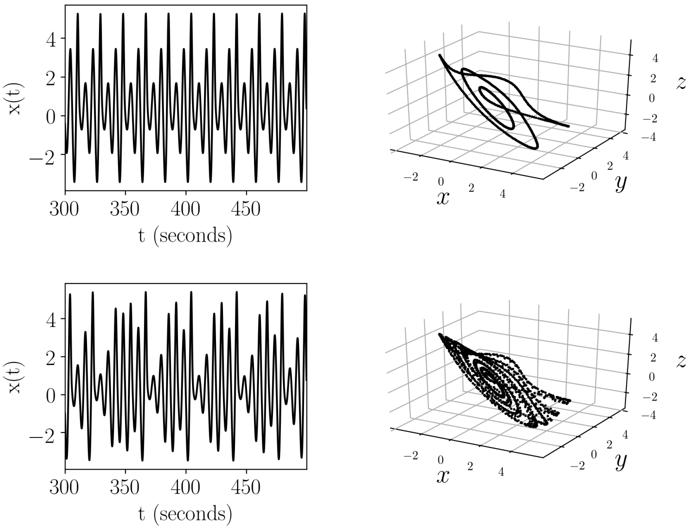
Choose τ to be first minimum of $I_\epsilon(\tau)$

False Nearest Neighbors

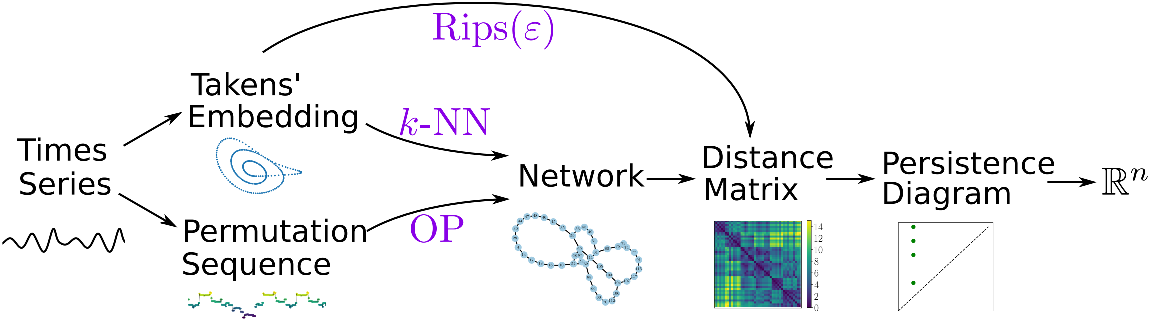


Choose d to be smallest when FNN is below some threshold.

Distinguish periodic vs chaotic



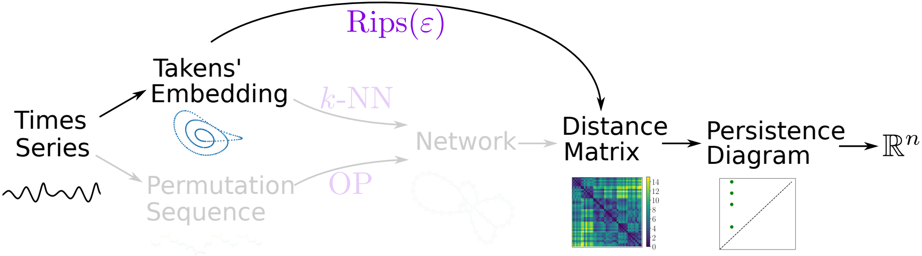
From Time Series to Scores



Section 1

Rips Complex from Takens Embedding

Pipeline



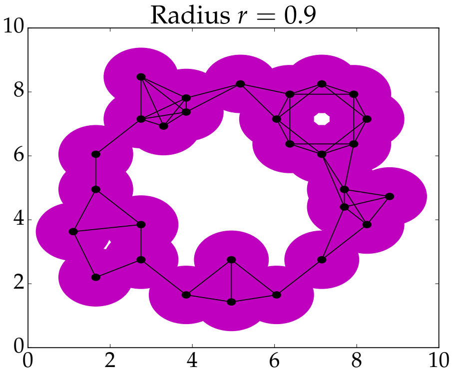
Rips(ε)
Takens' Embedding
Times Series
k-NN
Network
Permutation Sequence
OP
Distance Matrix
14
12
10
8
6
4
2
0
Persistence Diagram
$\mathbb{R}^n$

Persistence on point clouds: Rips complex

Definition

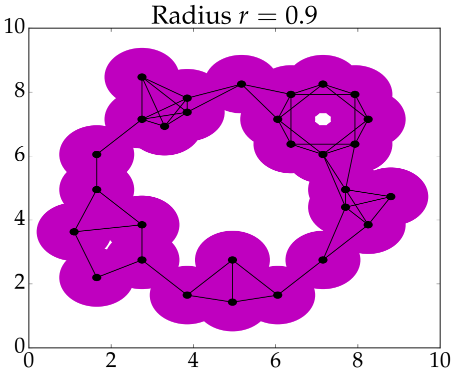
Given a point cloud $\chi \subseteq \mathbb{R}^d$,

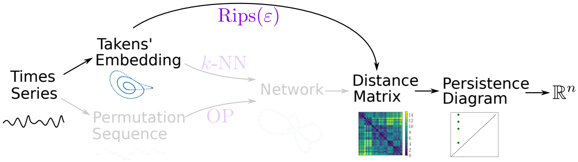
$$\mathrm{Rips}(\varepsilon) = \{\sigma \subseteq \chi \mid \|u - v\| \leq \varepsilon \ \forall u, v \in \sigma\}$$



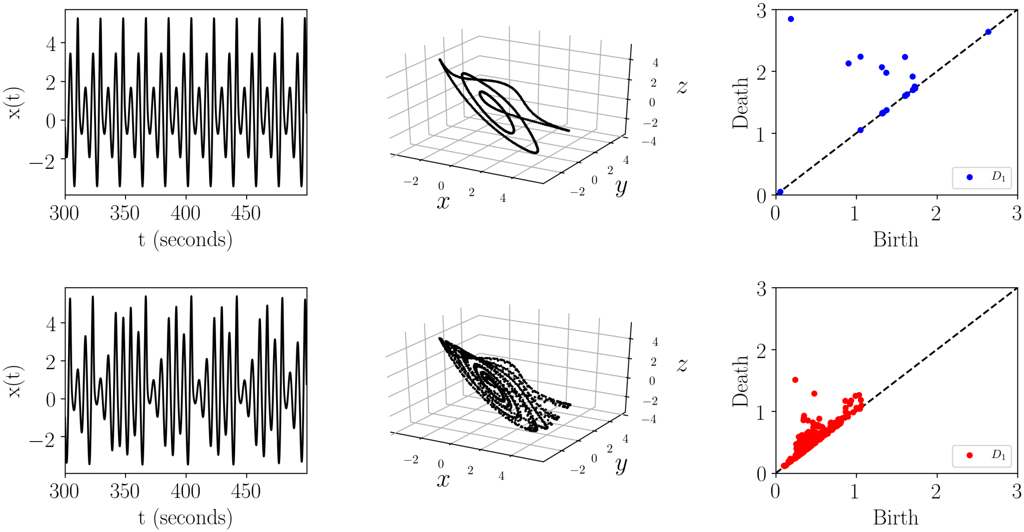
Method 1: Rips complex of Takens' embedding

- Fix parameters d, τ
- Construct point cloud
- Compute distance matrix in $\mathbb{R}^d$
- Compute persistence diagram

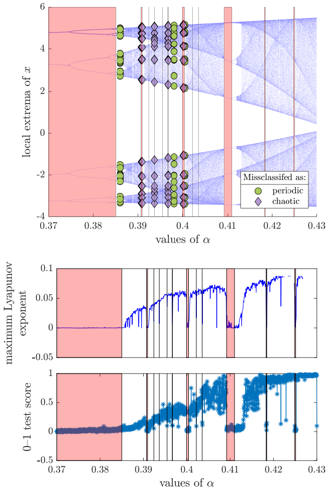


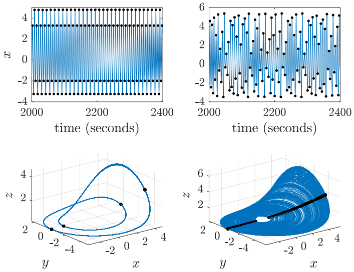


Example of Rips complex pipeline on Rossler



Rossler classification with tent functions

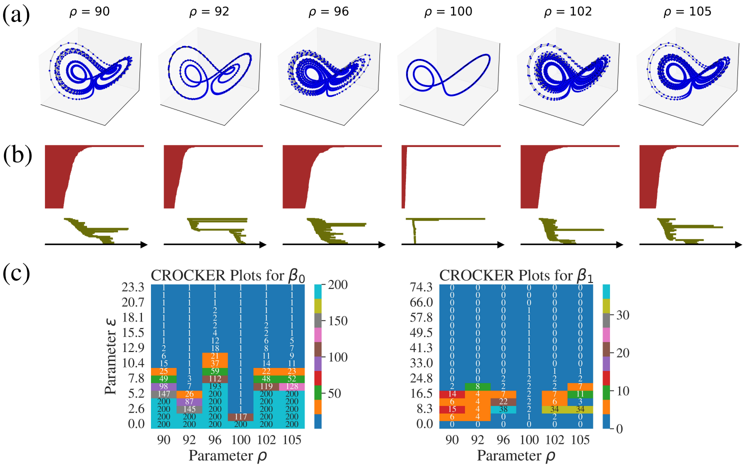




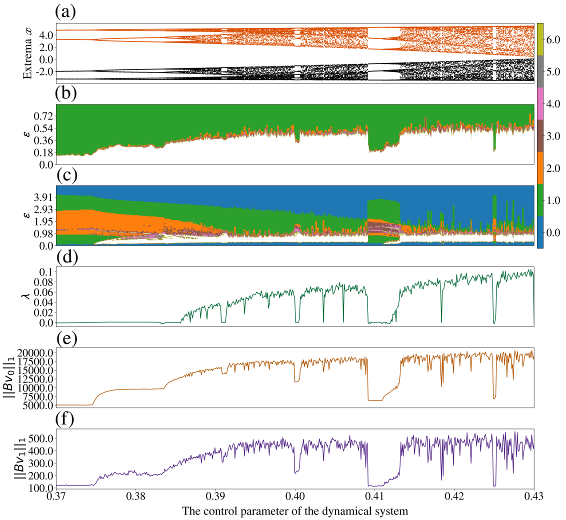
1. Use Lyapunov to decide on ground truth
2. Supervised learning on the template features
3. Misclassified labeled with circles/diamonds

Perea, Munch, Khasawneh. *Approximating Continuous Functions on Persistence Diagrams Using Template Functions*, arXiv:1902.07190, 2019.

Recall: CROCKER

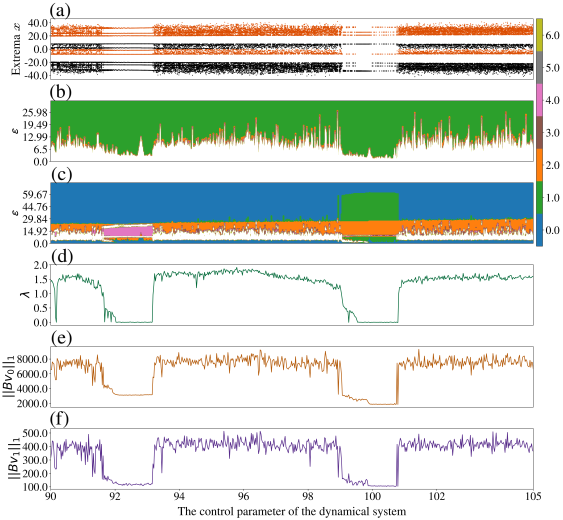


Crocker: Lorenz



Detecting bifurcations in dynamical systems with CROCKER plots. İsmail Güzel, Elizabeth Munch, and Firas A. Khasawneh. Chaos 2022.

Crocker: Rössler

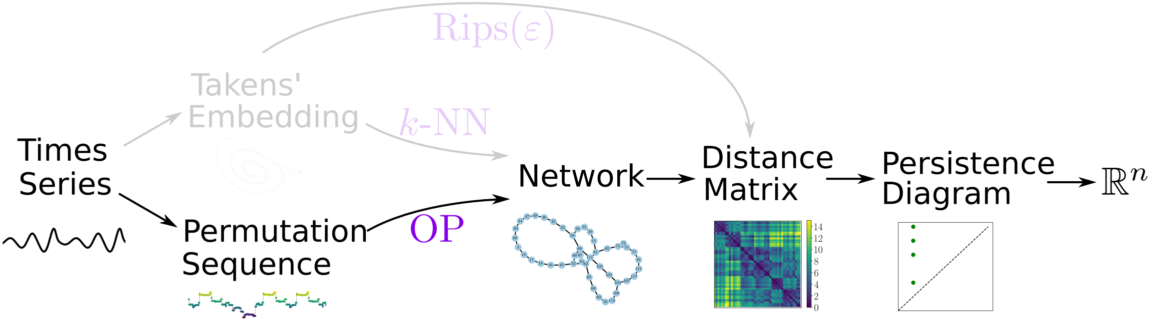


Detecting bifurcations in dynamical systems with CROCKER plots. İsmail Güzel, Elizabeth Munch, and Firas A. Khasawneh. Chaos 2022.

Section 2

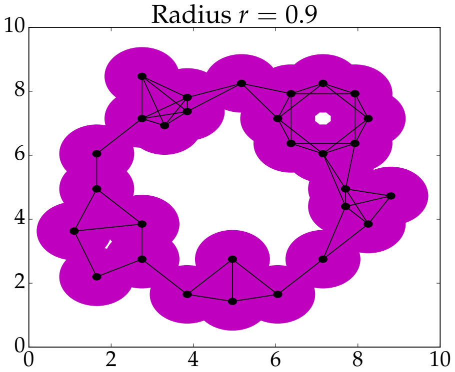
Graph-based methods from Takens Embedding

Second pipeline

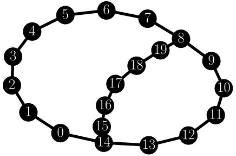


Inducing a filtration from a distance

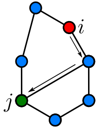
1. Given a distance $d : V \times V \to \mathbb{R}_{\geq 0}$.
2. K is complete simplicial complex with vertices V
3. $f : K \to \mathbb{R}_{\geq 0}$ given by $\sigma \mapsto \max_{\{u,v\} \subseteq \sigma} \{d(u,v)\}$
4. $K_a = \{\sigma \in K \mid f(\sigma) \leq a\}$
5. Filtration $K_{a_1} \subseteq K_{a_2} \subseteq \cdots \subseteq K_{a_k}$

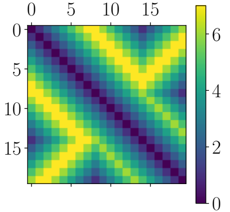


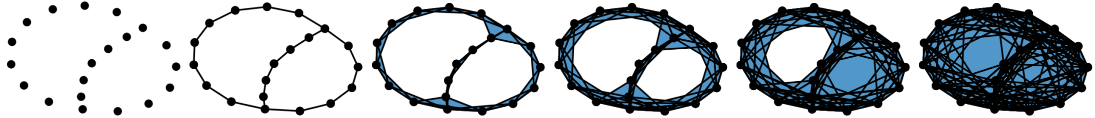
Persistence on graph



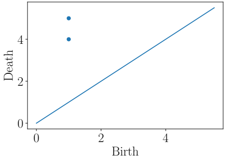
$d(u, v)$ = shortest path length







K_0 K_1 K_2 K_3 K_4 K_5

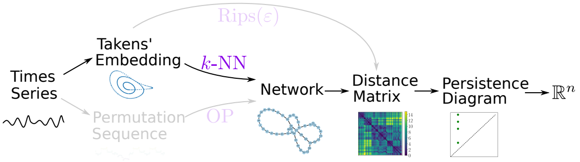


Method 2: k-NN graph of Takens' embedding

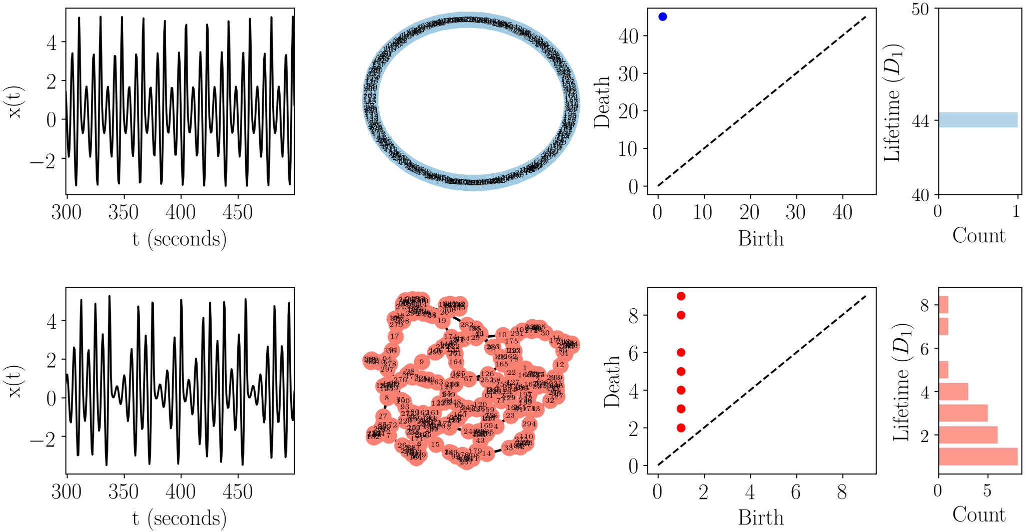
Definition

The (undirected, unweighted) k-NN graph of a point cloud χ has vertex set $V \cong \chi$ and an edge uv if $v \in \chi$ is among the k closest neighbors of v.

- Fix parameters d, τ
- Construct point cloud
- Choose k
- Construct k-NN Graph
- Compute shortest path distance in the graph
- Compute persistence diagram



k-NN graph of Takens' embedding example



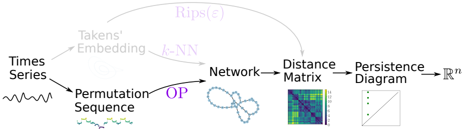
Method 2: Ordinal partition network

Small 2013; McCullough, Small, Stemler, Iu 2015

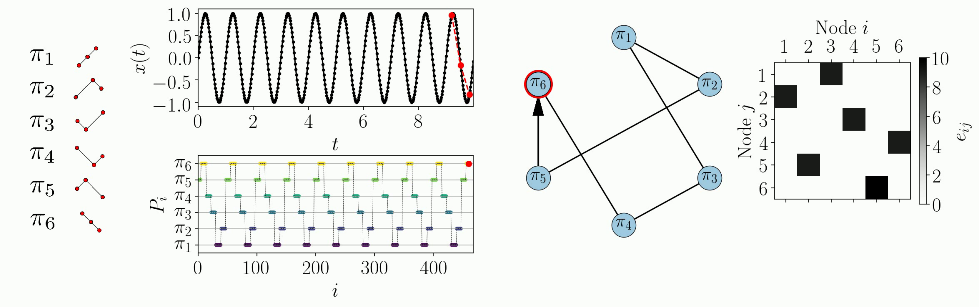
- Fix parameters d, τ
- Construct point cloud
- Construct ordinal partition network
- Compute shortest path distance in the graph
- Compute persistence diagram

Definition

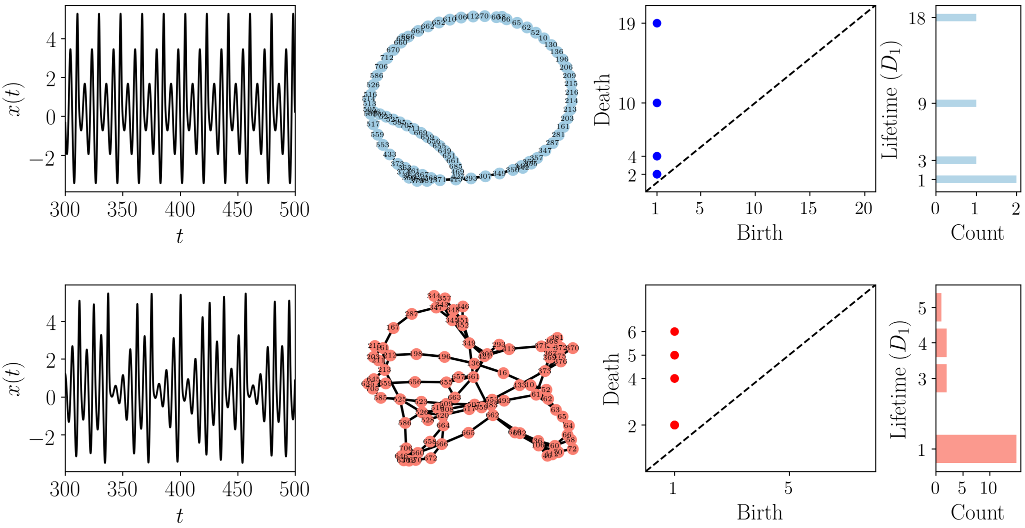
Given a point in χ, $(x_1, \cdots, x_d)$, the associated permutation π is the permutation of the set $\{1, \cdots, d\}$ that satisfies $x_{\pi(1)} \leq x_{\pi(2)} \leq \cdots \leq x_{\pi(d)}$.



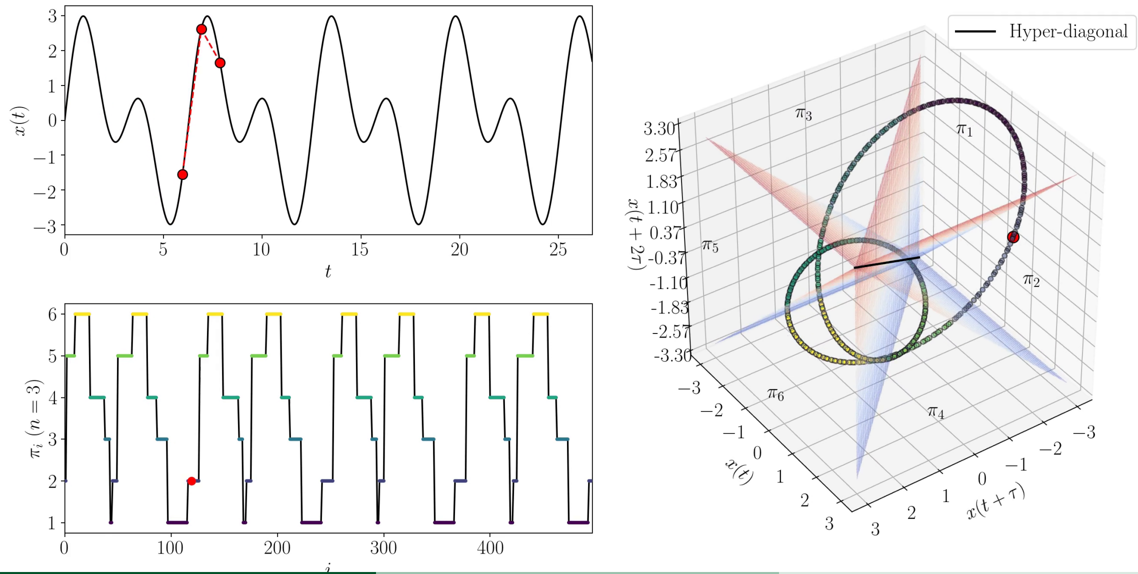
Ordinal Partition Network Example



Ordinal partition network of Rossler examples



Relationship to Takens Embedding



Persistent entropy

Definition (Chintakunta et al. 2015)

The persistent entropy of a diagram D is

$$E(D) = -\sum_{x \in D} \frac{\mathrm{pers}(x)}{\mathcal{L}(D)} \log_2 \left(\frac{\mathrm{pers}(x)}{\mathcal{L}(D)} \right),$$

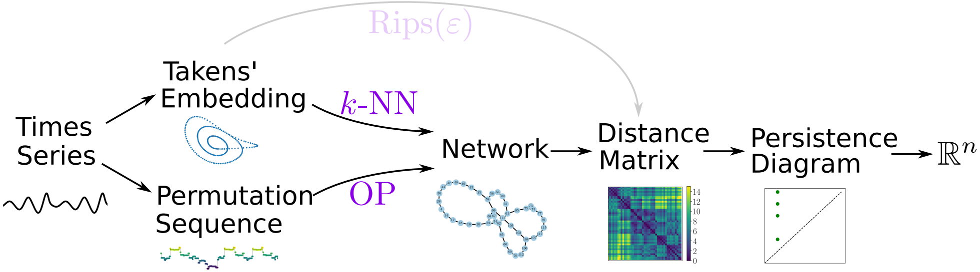
where $\mathrm{pers}(x_1, x_2) = |x_2 - x_1|$ and $\mathcal{L}(D) = \sum_{x \in D} \mathrm{pers}(x)$.

Definition

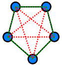
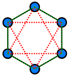
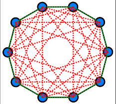
The normalized persistent entropy of a diagram D is

$$E'(D) = \frac{E(D)}{\log_2 \left(\mathcal{L}(D) \right)}.$$

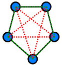
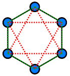
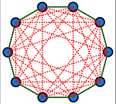
Pipeline


Rips(ε)
Takens'
Embedding
Times
Series
Permutation
Sequence
k-NN
OP
Network
Distance
Matrix
Persistence
Diagram
ℝⁿ

Periodicity score

n	Cycle	r_B	r_D	L_n
3		1	1	0
4		1	2	1
5		1	2	1
6		1	2	1
	⋮			
10		1	4	3
	⋮			

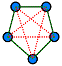
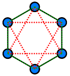
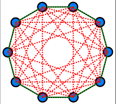
Periodicity score

n	Cycle	r_B	r_D	L_n
3		1	1	0
4		1	2	1
5		1	2	1
6		1	2	1
	⋮			
10		1	4	3
	⋮			

Persistence diagram has 1 point at $(1, \lceil \frac{n}{3} \rceil)$.
Lifetime is

$$L_n = \mathrm{maxpers}(D') = \left\lceil \frac{n}{3} \right\rceil - 1.$$

Periodicity score

n	Cycle	r_B	r_D	L_n
3		1	1	0
4		1	2	1
5		1	2	1
6		1	2	1
	⋮			
10		1	4	3
	⋮			

Persistence diagram has 1 point at $(1, \lceil \frac{n}{3} \rceil)$.
Lifetime is

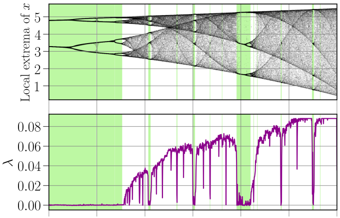
$$L_n = \mathrm{maxpers}(D') = \left\lceil \frac{n}{3} \right\rceil - 1.$$

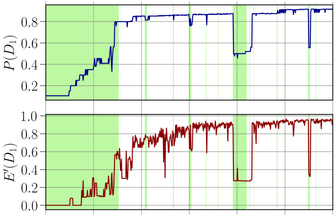
Definition

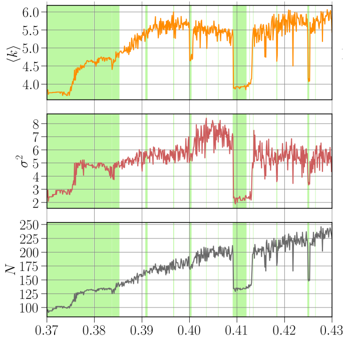
The network periodicity score is

$$P(D) = 1 - \frac{\mathrm{maxpers}(D)}{L_n}.$$

Ordinal Partition Network Results







$P(D)$: periodicity score

$E'(D)$: normalized persistent entropy

$< k >$: mean out degree

σ^2: variance of out degree

N: number of nodes

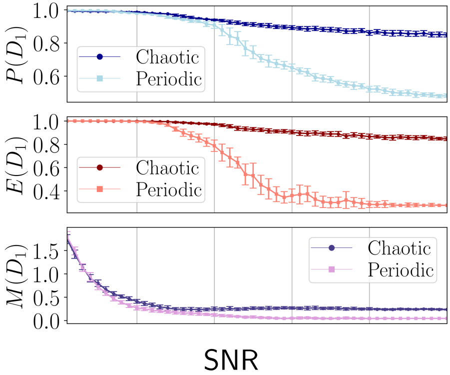
Myers, Munch, Khasawneh. *Persistent Homology of Complex Networks for Dynamic State Detection*, Phys. Rev. E, 2019.

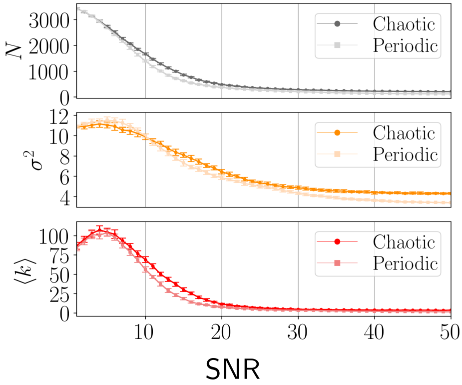
Results on other systems

System/ Data	k-NN Graph from Takens' Embedding						Ordinal Partition Graph					
	$E'(D_1)$		$M(D_1)$		$P(D_1)$		$E'(D_1)$		$M(D_1)$		$P(D_1)$	
	Per.	**Ch.**	**Per.**	**Ch.**	**Per.**	**Ch.**	**Per.**	**Ch.**	**Per.**	**Ch.**	**Per.**	**Ch.**
Chua Circuit	0.00	0.80	0.001	0.19	0.54	0.89	0.21	0.72	0.051	0.19	0.42	0.88
Lorenz	0.04	0.84	0.005	0.16	0.64	0.93	0.18	0.95	0.026	0.36	0.28	0.96
Rossler	0.00	0.85	0.001	0.18	0.50	0.94	0.00	0.89	0.036	0.28	0.33	0.85
Coupled Lorenz-Rossler	0.00	0.82	0.003	0.16	0.46	0.94	0.00	0.87	0.033	0.35	0.56	0.92
Bi-directional Rossler	0.00	0.76	0.004	0.13	0.55	0.87	0.25	0.91	0.064	0.29	0.40	0.92
Mackey-Glass	0.00	0.67	0.001	0.07	0.56	0.93	0.30	0.96	0.077	0.37	0.25	0.93

Myers, Munch, Khasawneh. *Persistent Homology of Complex Networks for Dynamic State Detection*, Phys. Rev. E, 2019.

Resilience to additive noise





Myers, Munch, Khasawneh. *Persistent Homology of Complex Networks for Dynamic State Detection*, Phys. Rev. E, 2019.

Section 3

Zigzags and the BuZZ method

Zigzag persistent homology

- Given a collection of simplicial complexes with inclusions **that go in different directions**,

$$K_0 \hookrightarrow K_1 \hookleftarrow K_2 \hookrightarrow \cdots \hookleftarrow K_n$$

  then the inclusion maps induce maps on the p-th homology groups **that go in different directions**,

$$H_p(K_0) \to H_p(K_1) \leftarrow H_p(K_2) \to \cdots \leftarrow H_p(K_n).$$

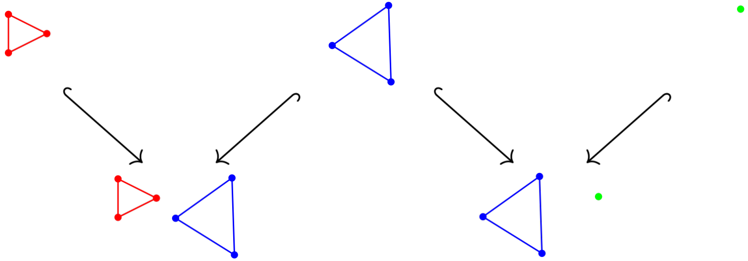
- This a sequence of vector spaces and linear maps called a **zigzag module**.

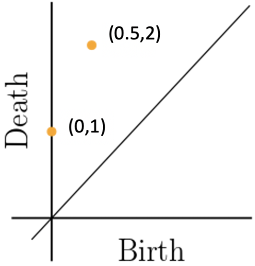
G. Carlsson and V. de Silva. Zigzag persistence. Foundations of Computational Mathematics, 10(4):367-405, 2010.

Zigzag persistence

- For a zigzag module $\mathbb{V}$, the **zigzag persistence** of $\mathbb{V}$ is the multiset of birth, death pairs

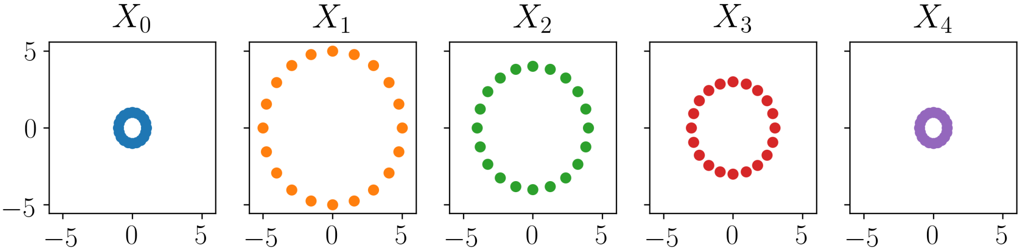
$$Pers(\mathbb{V}) = \{[b_j, d_j) \subset \{1, \ldots, n\} \mid j = 1, \ldots N\}.$$





Motivation

Using zigzag persistence, can we detect the range of point clouds that are "most circular"?

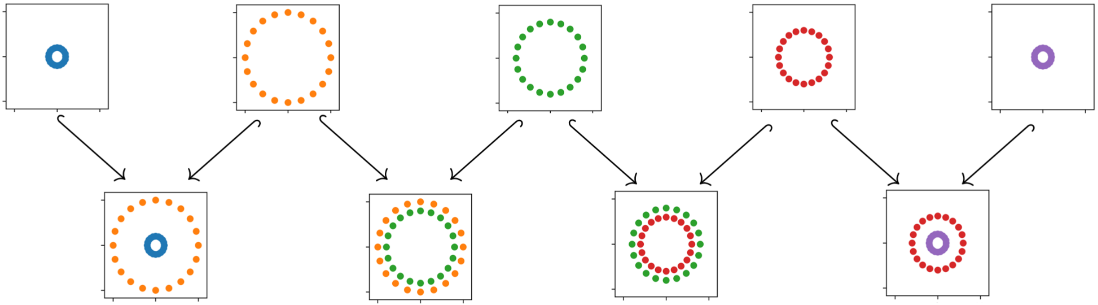


Setting up the zigzag

$$
\begin{array}{ccccccccccc}
X_0 & & X_1 & & X_2 & \cdots & X_{n-1} & & X_n \\
& \searrow & & \swarrow \; \searrow & & \swarrow & & \searrow & & \swarrow \\
& X_0 \cup X_1 & & & X_1 \cup X_2 & & & & X_{n-1} \cup X_n.
\end{array}
$$

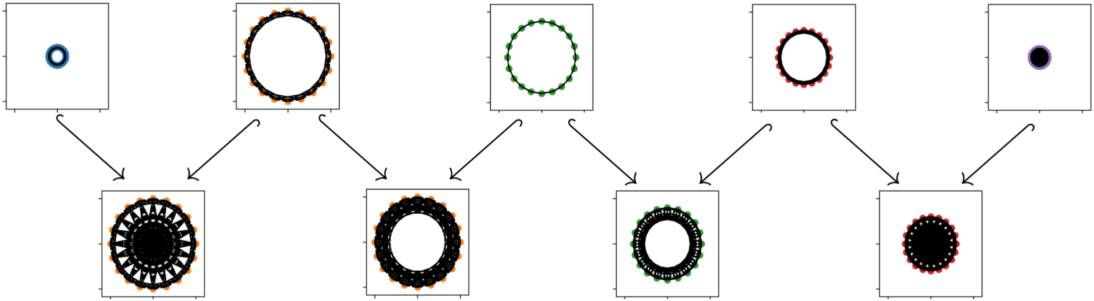
Setting up the zigzag

$$
\begin{array}{ccccccccc}
X_0 & & X_1 & & X_2 & \cdots & X_{n-1} & & X_n \\
& \searrow & & \swarrow \quad \searrow & & \swarrow & & \searrow \quad \swarrow & \\
& X_0 \cup X_1 & & X_1 \cup X_2 & & & & X_{n-1} \cup X_n. &
\end{array}
$$

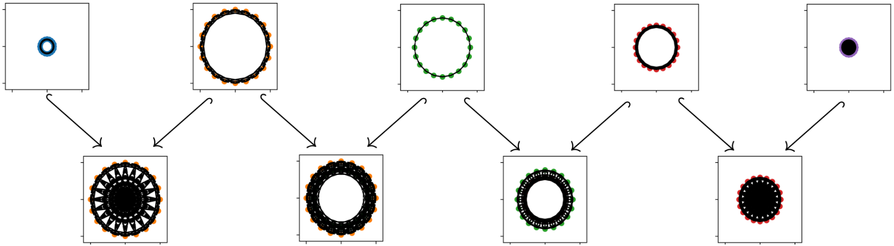


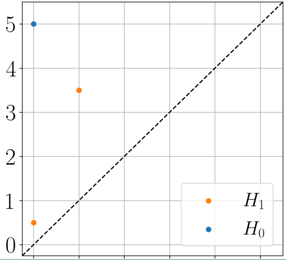
Zigzag diagram of Rips complexes

$$\begin{array}{ccccccccc} R(X_0,r) & & R(X_1,r) & & R(X_2,r) & \cdots & R(X_{n-1},r) & & R(X_n,r) \\ & \searrow & & \swarrow \quad \searrow & & \swarrow & & \searrow \quad \swarrow & \\ & R(X_0 \cup X_1, r) & & R(X_1 \cup X_2, r) & & & & R(X_{n-1} \cup X_n, r). & \end{array}$$

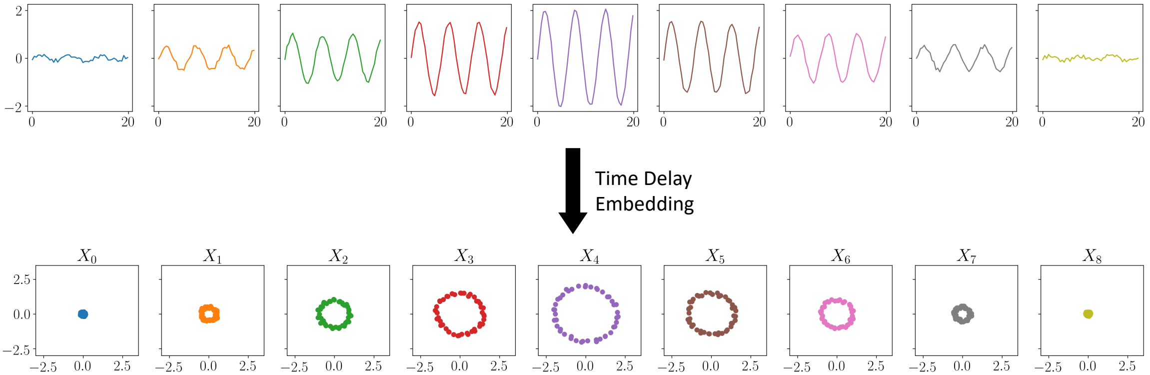


How to interpret the diagrams

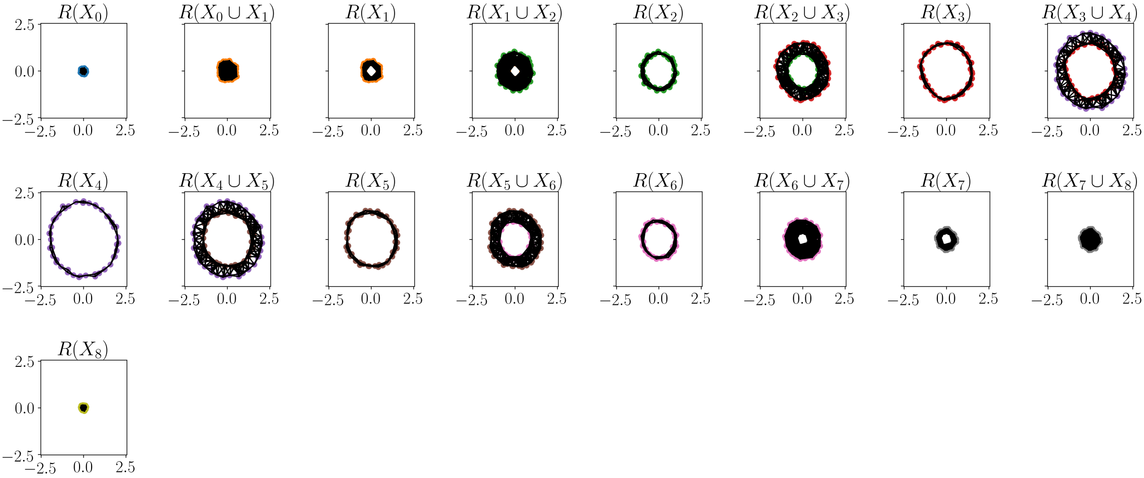




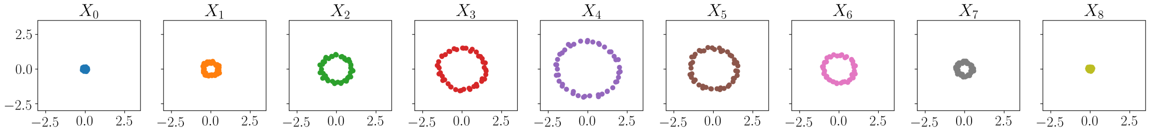
Time series to point clouds

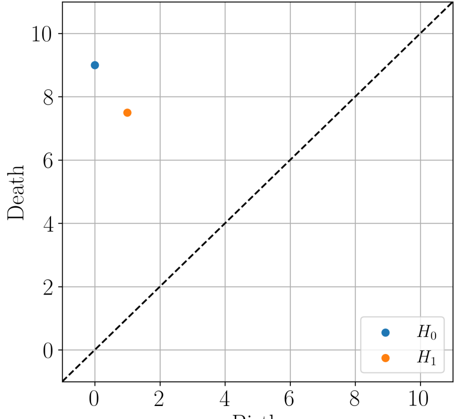


Setting up the Zigzag



Result



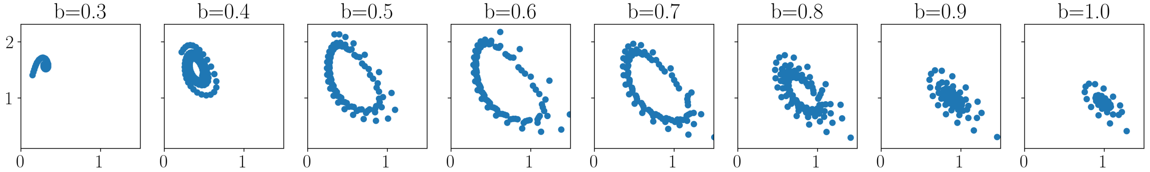


Can we detect changes in behavior of this dynamical system?

Sel'kov model for glycolysis:

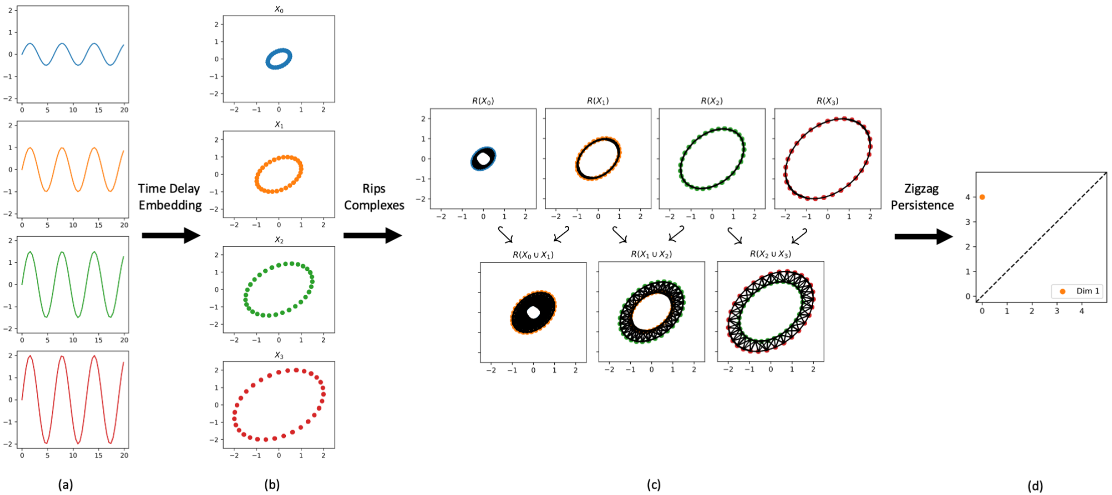
$$\frac{dx}{dt} = -x + ay + x^2 y \qquad \frac{dy}{dt} = b - ay - x^2 y$$

Specifically, can we detect for which values of b there is a Hopf bifurcation?

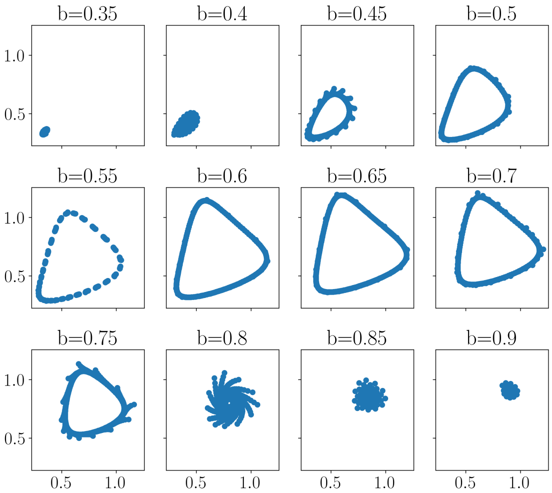


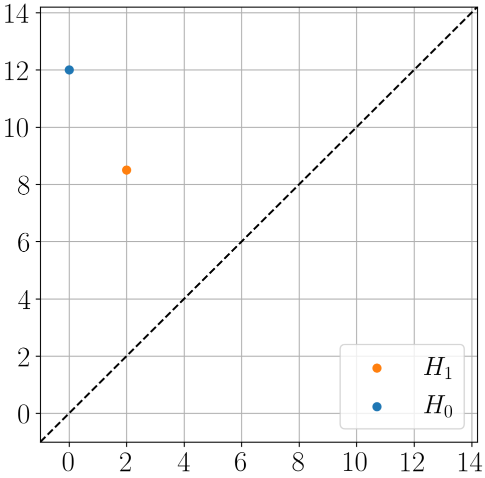
E. Sel'Kov. Self-oscillations in glycolysis. European Journal of Biochemistry, 4(1):79-86, 1968.

Overview of the Bifurcations using ZigZag (BuZZ) Method

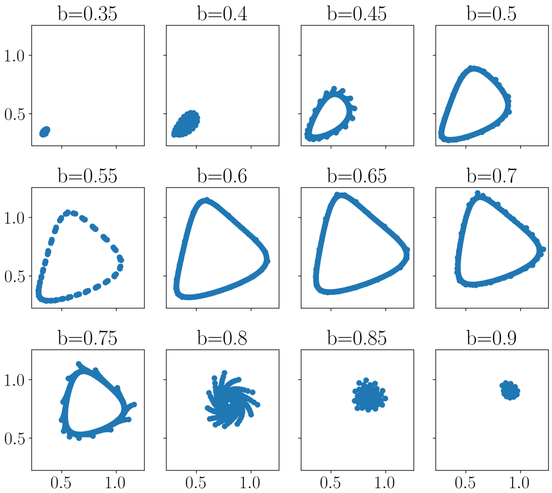


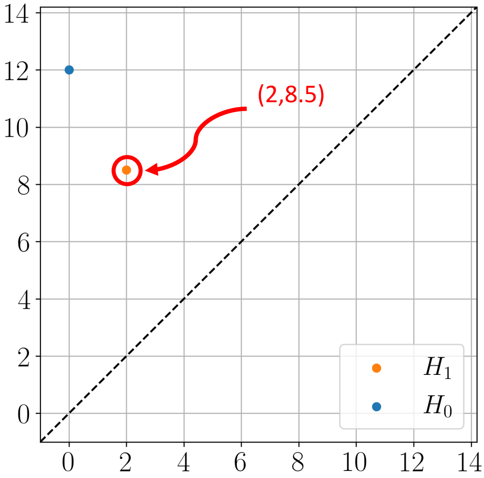
The result


b=0.35
b=0.4
b=0.45
b=0.5
b=0.55
b=0.6
b=0.65
b=0.7
b=0.75
b=0.8
b=0.85
b=0.9
1.0
0.5
0.5
1.0

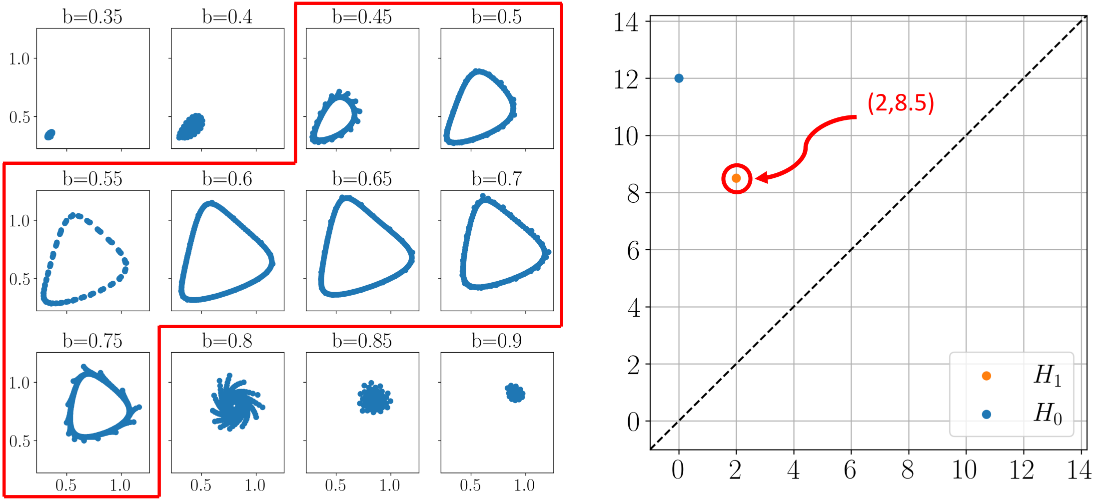

0
2
4
6
8
10
12
14
H_1
H_0

The result


b=0.35
b=0.4
b=0.45
b=0.5
b=0.55
b=0.6
b=0.65
b=0.7
b=0.75
b=0.8
b=0.85
b=0.9
1.0
0.5
0.5
1.0


(2,8.5)
H_1
H_0
0
2
4
6
8
10
12
14

The result



b=0.35
b=0.4
b=0.45
b=0.5
b=0.55
b=0.6
b=0.65
b=0.7
b=0.75
b=0.8
b=0.85
b=0.9
1.0
0.5
0.5
1.0
(2,8.5)
H_1
H_0
0
2
4
6
8
10
12
14

We're done!