More Reeb Graphs

Lecture 17 - CMSE 890

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Check-in

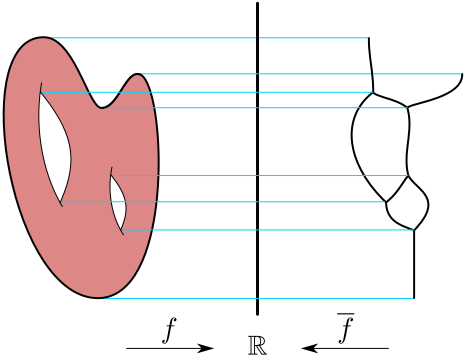
Goals for today:

- 7.1 More Reeb Graph Definitions
- 7.3 Reeb Graph Metrics

Section 1

Reeb graphs

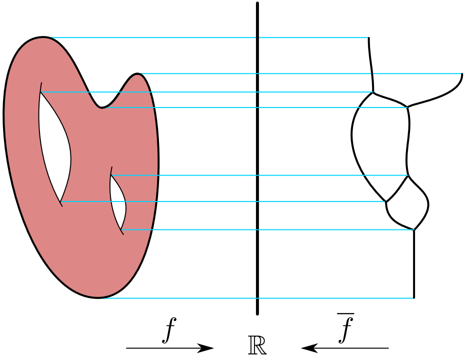
Reeb graph definition



Given a function $f : X \to \mathbb{R}$.
Define an equivalence relation $\sim$ by $s \sim y$ iff

- $f(x) = f(y) = \alpha$
- x and y are in the same connected component of the level set $f^{-1}(\alpha)$.
- Let $[x]$ denote the equivalence class of $x \in X$.
- The Reeb graph R_f of $f : X \to \mathbb{R}$ is the quotient space $X/\sim$.
- Let $\Phi : X \to R_f$, $x \to [x]$ be the quotient map.

Induced map $\tilde{f} : R_f \to \mathbb{R}$



Lazy notation: Just write $f : R_f \to \mathbb{R}$

What I mean by a Reeb graph....

Discrete vs continuous viewpoints

Quotient Space $X/\sim$ with function $\overline{f} : [a] \to f(a)$

Graph (V, E) with function $f : V \to \mathbb{R}$

- For any edge uv, $f(u) \neq f(v)$

Up and down degree

Given a node $x \in V$ in the vertex set $V := V(R_f)$ of the Reeb graph R_f,

- up-degree of x is the number of edges (xu) incident to x with $f(u) > f(x)$
- down-degree of x is the number of edges (xu) incident to x with $f(u) < f(x)$

Regular vs critical node

A node is

- regular if both of its up-degree and down-degree equal to 1
- critical otherwise.

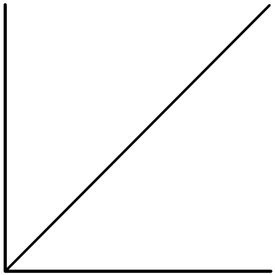
Types of critical points

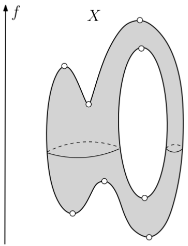
A critical point is a

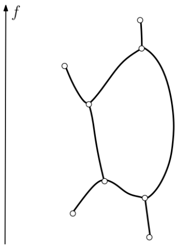
- a minimum if it has down-degree 0
- a maximum if it has up-degree 0
- a down-fork if it has down-degree larger than 1
- an up-fork if it has up-degree larger than 1

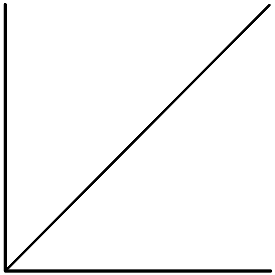
Warning: a critical point can be denegnerate, i.e. have more that one type of criticality

Comparing the sublevelset persistence diagrams









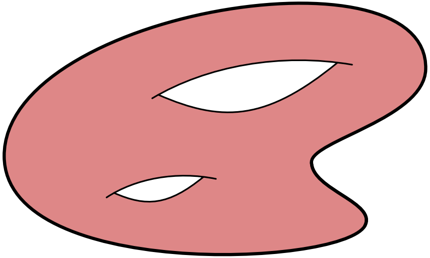
Relationship between Betti numbers

Theorem

For a tame function $f : X \to \mathbb{R}$, $\beta_0(X) = \beta_0(R_f)$ *and* $\beta_1(X) \geq \beta_1(R_f)$.

A note on contractible X

A Reeb graph is dependent on the function



What if X is a 2-manifold

Theorem

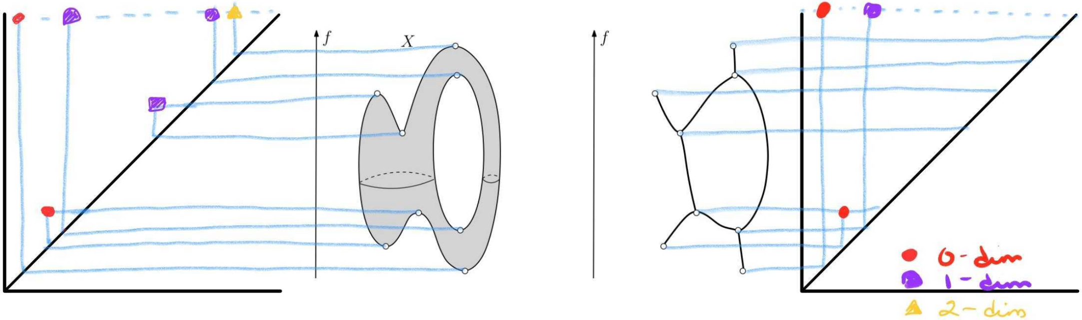
Let $f : X \to \mathbb{R}$ be a Morse function on a connected, compact 2-manifold.

- *If X is orientable, $\beta_1(R_f) = \beta_1(X)/2$*
- *If X is not orientable, $\beta_1(R_f) \leq \beta_1(X)/2$*

Note that this is independent of the function!

Back to that example

Unsatisfying loss of information



Section 2

Reeb Graph Metrics

Distances Between Reeb Graphs

Definition

A metric on a set M is a function $d : M \times M \to \mathbb{R}_{\geq 0}$ such that

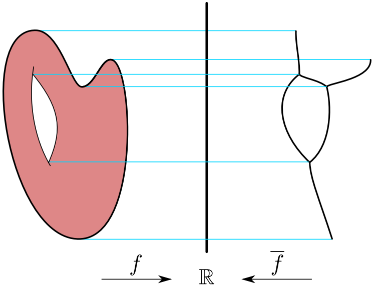
- $d(x, y) \geq 0$ and
  $d(x, y) = 0$ iff $x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

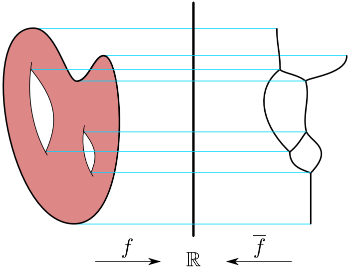
Reeb graph metrics

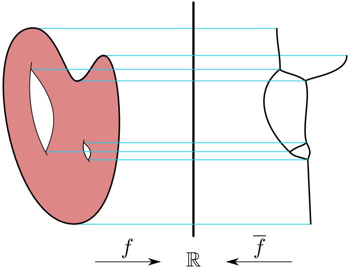
Goal

Create and study Reeb graph metrics:

- $d((G, f), (G', f')) = 0$ iff $G \cong G'$ and $f = f'$







Metrics for Reeb graphs

- d_I interleaving
- d_{FD} functional distortion
- d_G edit
- d_B bottleneck

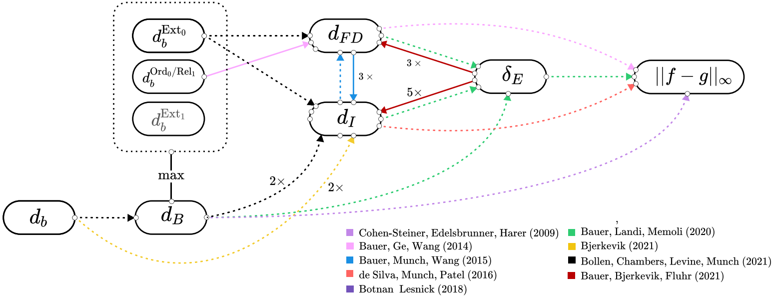


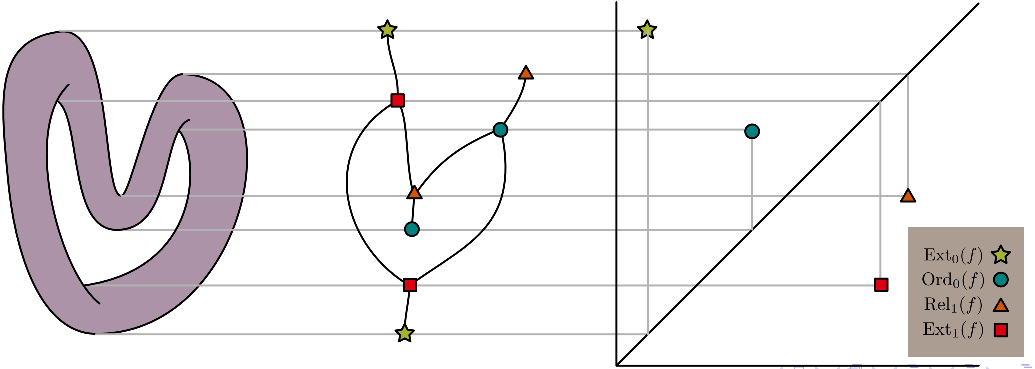
Figure inspired by U. Bauer's talk, SoCG 2020;
Drawn by Brian Bollen, arXiv:2110.05631

Section 3

Bottleneck distance

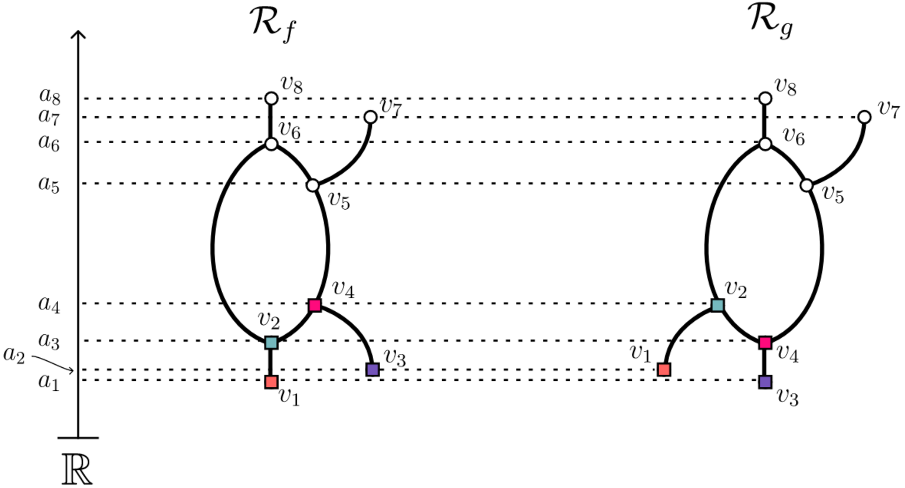
Represent your Reeb graph as its extended persistence diagram
Compute (label preserving) bottleneck distance

$$d_B(R_f, R_g) := d_B(Dgm(R_f), Dgm(R_g))$$



Not a perfect representation

Two Reeb graphs with the same extended persistence diagram



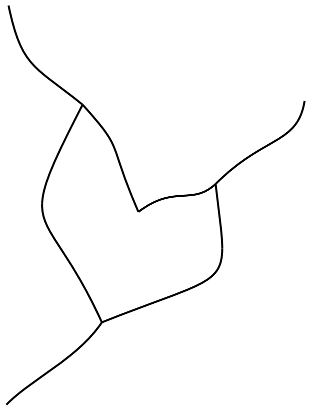
Pros & Cons

Section 4

Interleaving Distance

Smoothing Reeb Graphs: S_ε

Definition by example



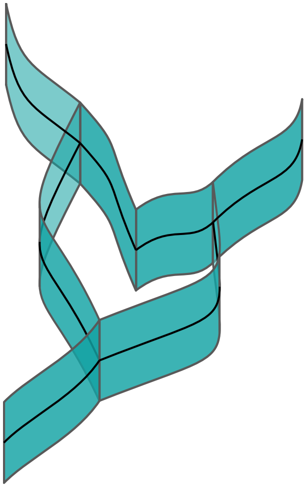
- Given Reeb graph (G, f)
- Thicken:

$$\begin{aligned} f_\varepsilon : G \times [-\varepsilon, \varepsilon] &\longrightarrow \mathbb{R} \\ (x,t) &\longmapsto f(x) + t \end{aligned}$$

- Take Reeb graph of $(G \times [-\varepsilon, \varepsilon], f_\varepsilon)$

Smoothing Reeb Graphs: S_ε

Definition by example



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Smoothing Reeb Graphs: S_ε

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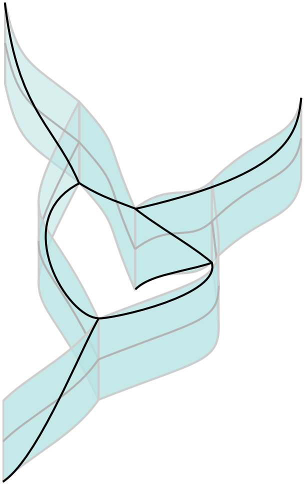

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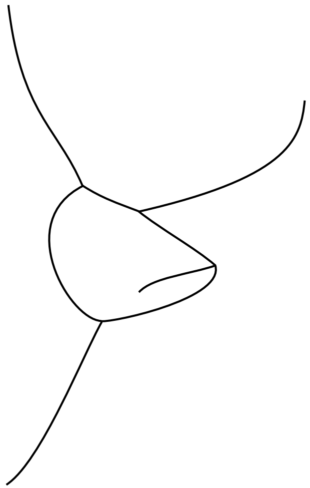

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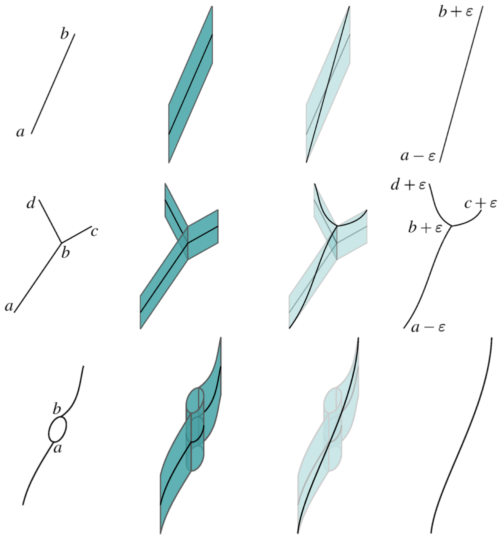
- Take Reeb graph of $(G \times [-\varepsilon, \varepsilon], f_\varepsilon)$

More Smoothing Examples



Smoothing definition

Definition

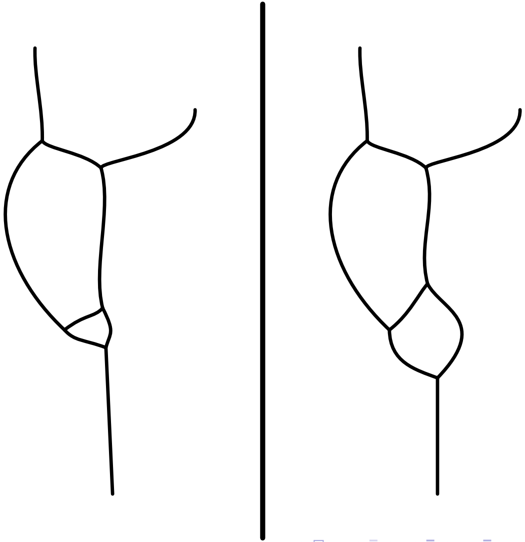
The ε-smoothed Reeb graph of (G, f) is the Reeb graph of $(G \times [-\varepsilon, \varepsilon], f_\varepsilon)$

$$\begin{tikzcd}
& (G \times [-\varepsilon,\varepsilon], f_\varepsilon) \arrow[dr, "q"] & \\
(G, f) \arrow[ur, hook, "(\mathrm{Id}_G{,}0)"] \arrow[rr, "\eta"] & & S_\varepsilon(G, f)
\end{tikzcd}$$

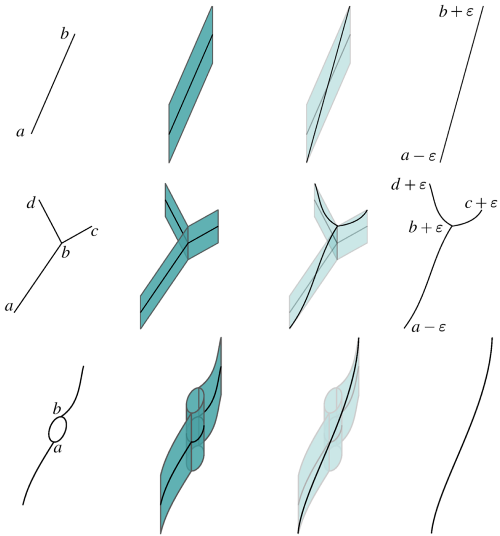
Function Preserving Maps

Definition

A morphism $\varphi : (G, f) \to (H, g)$ is a map $\varphi : G \to H$ which is function preserving ($f = g \circ \varphi$).



Smoothings Automatically Come With Morphisms



Reeb-interleaving

Definition

Let $(G,f),(H,g)$ be given.
An ε-interleaving consists of two morphisms

$$\varphi\colon (G,f) \to S_\varepsilon(H,g);\ \psi\colon (H,g) \to S_\varepsilon(G,f)$$

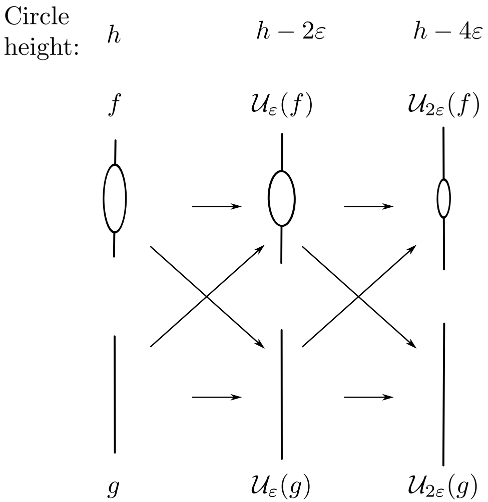
such that

$$\begin{tikzcd}
(G,f) \arrow[rr] \arrow[drr, "\varphi" near start] & & S_\varepsilon(G,f) \arrow[rr] \arrow[drr, dashed, "S_\varepsilon[\varphi]" near start] & & S_{2\varepsilon}(G,f) \\
(H,g) \arrow[rr] \arrow[urr, "\psi"' near start] & & S_\varepsilon(H,g) \arrow[rr] \arrow[urr, dashed, "S_\varepsilon[\psi]"' near start] & & S_{2\varepsilon}(H,g)
\end{tikzcd}$$

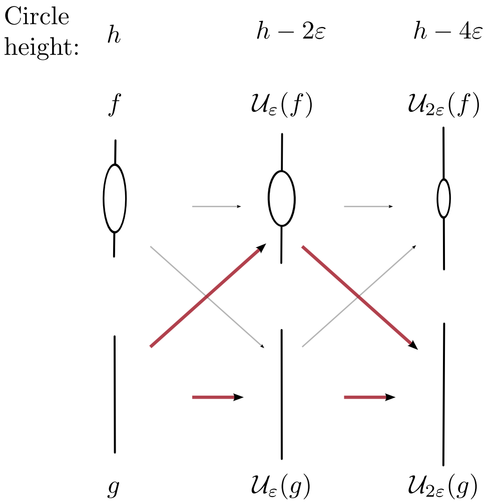
commutes. The interleaving distance is defined to be

$$d_I((G,f),(H,g)) = \inf\{\varepsilon \mid (G,f) \text{ and } (H,g) \text{ are } \varepsilon\text{-interleaved}\}.$$

Building an interleaving



Building an interleaving



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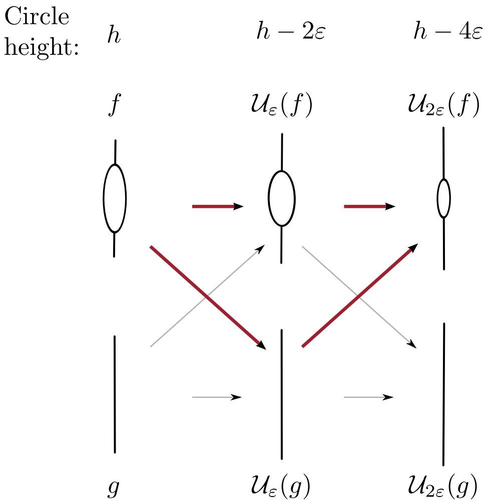
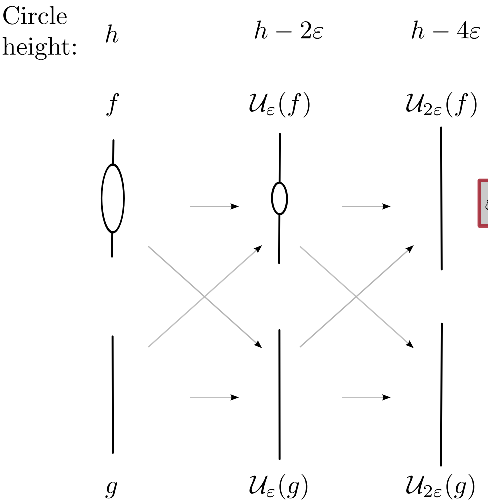


Building an interleaving



$\boxed{\varepsilon \geq h/4}$

Properties

Theorem (de Silva, EM, Patel 2016)

The interleaving distance is an **extended metric***.*

$$d_I((G, f), (H, g)) < \infty$$
$$\Leftrightarrow$$
$$\beta_0(G) = \beta_0(H)$$

Properties

Theorem (de Silva, EM, Patel 2016)

The interleaving distance is an **extended metric***.*

$$d_I((G,f),(H,g)) < \infty$$
$$\Leftrightarrow$$
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Theorem (dS, EM, P 2016)

Given $f, g : G \to \mathbb{R}$, the interleaving distance is **stable***, i.e.*

$$d_I(\mathcal{R}(G,f), \mathcal{R}(G,g)) \leq \|f-g\|_\infty.$$

Pros & Cons

Section 5

Functional Distortion Distance

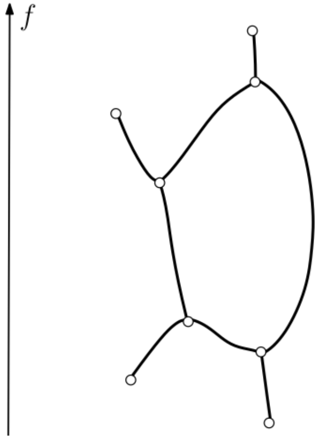
Distance in the Reeb graph

Definition

Let $u, v \in \mathcal{R}_f$ (not necessarily vertices) and let π be a continuous path between u and v, denoted $u \rightsquigarrow v$.
$\mathrm{range}(\pi) = [\min_{x \in \pi} f(x), \max_{x \in \pi} f(x)]$.
$\mathrm{height}(\pi) = \max_{x \in \pi} f(x) - \min_{x \in \pi} f(x)$
Distance between u and v:

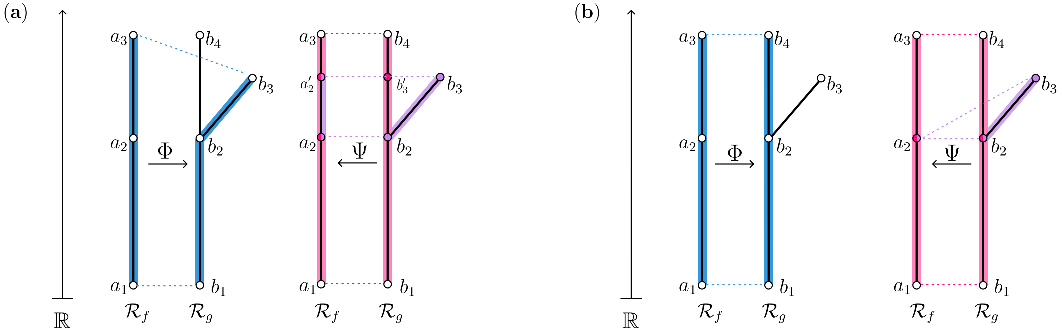
$$d_f(u, v) = \min_{\pi : u \rightsquigarrow v} \mathrm{height}(\pi),$$

where π ranges over all continuous paths from u to v.



Continuous maps

$\Phi : \mathcal{R}_f \to \mathcal{R}_g$, $\Psi : \mathcal{R}_f \to \mathcal{R}_g$ continuous maps (not necessarily function preserving).

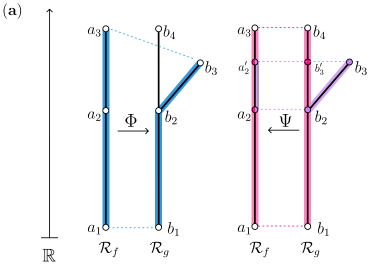


$$G(\Phi, \Psi) = \{(x, \Phi(x)) \mid x \in \mathcal{R}_f\} \cup \{(\Psi(y), y) \mid y \in \mathcal{R}_g\}$$

Definition

The **point distortion** λ between $(x, y), (x', y') \in G(\Phi, \Psi)$ is defined as

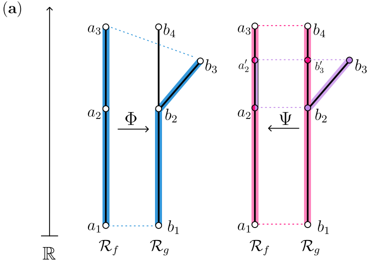
$$\lambda((x, y), (x', y')) = \frac{1}{2}|d_f(x, x') - d_g(y, y')|.$$



Definition

The **map distortion** $D(\Phi, \Psi)$ between $\mathcal{R}_f$ and $\mathcal{R}_g$ is the supremum of point distortions ranging over all possible pairs in the supergraph $G(\Phi, \Psi)$. That is,

$$D(\Phi, \Psi) = \sup_{(x,y),(x',y') \in G(\Phi,\Psi)} \lambda((x, y), (x', y')).$$



Definition

The **functional distortion distance** is defined as

$$d_{FD}(\mathcal{R}_f, \mathcal{R}_g) = \inf_{\Phi,\Psi} \max\{D(\Phi, \Psi), ||f - g \circ \Phi||_\infty, ||f \circ \Psi - g||_\infty\},$$

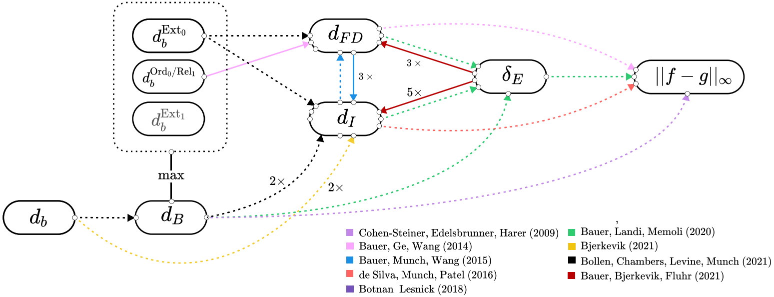
where Φ and Ψ range over all continuous maps between $\mathcal{R}_f$ and $\mathcal{R}_g$.

Pros & Cons

TL;DR

Metrics for Reeb graphs

- d_I interleaving
- d_{FD} functional distortion
- d_G edit
- d_B bottleneck



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