Persistence

Lecture 9 - CMSE 890

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Goals

Goals for today:

- Persistent homology

Need to do HW:

- Eugene
- Edem
- Jannik

Section 1

Last time

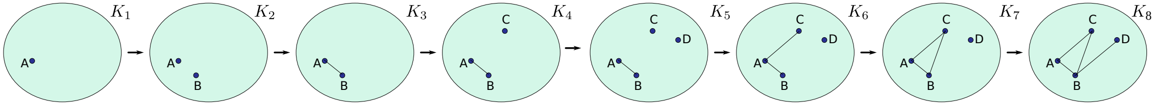
Last time: Simplicial Complex Filtration

Definition

A filtration $\mathcal{F} = \mathcal{F}(K)$ of a simplicial complex K is a nested sequence of its subcomplexes

$$\mathcal{F} : \emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K.$$

$\mathcal{F}$ is called *simplex-wise* if $K_i \setminus K_{i-1}$ is empty or a single simplex.

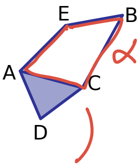


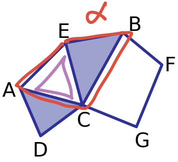
Betti curve

The p-th Betti curve is a function

$$\begin{aligned}\beta_p(\mathcal{F}) : \overset{\mathbb{R}}{\cancel{\mathbb{Z}}} &\to \mathbb{Z}\\ i &\mapsto \beta_p(K_i) = \mathrm{rk}(H_p(K_i))\end{aligned}$$

Induced map



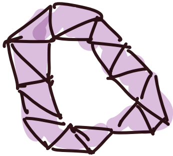


Given a simplicial map $f : K \to L$, the induced map on homology is defined by

$$\begin{array}{rccc} f_* : & H_p(K) & \to & H_p(L) \\ & [\alpha] & \mapsto & [f_\#(\alpha)] \end{array}$$

$[\alpha]$

$[\alpha] = [AE + EC + AC]$



Section 2

Persistence

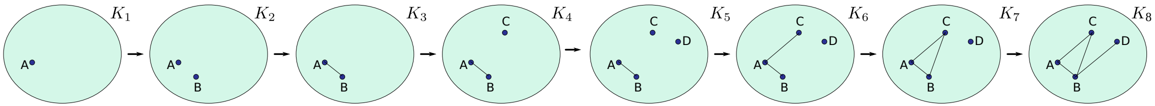
Persistence module

Given a filtration $\mathcal{F}$, the p-dimensional persistence module is

$$H_p(\mathcal{F}) : 0 = H_p(K_0) \to H_p(K_1) \to H_p(K_2) \cdots \to H_p(K_n) = H_p(K)$$

with maps $h_p^{i,j} : H_p(K_i) \to H_p(K_j)$ induced by inclusion.

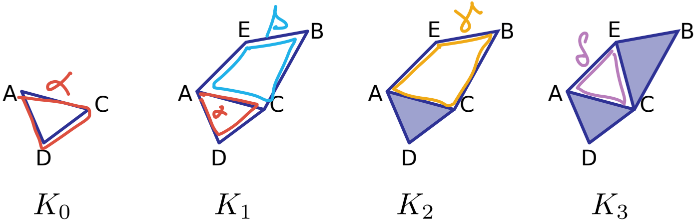
What's going on algebraically?



$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

$$\begin{tikzcd}
{[A]} \arrow[r, mapsto] & {[A]} \arrow[r, mapsto] & {[A]} \arrow[r, mapsto] \arrow[d, phantom, "\shortparallel"] & {[A]} \arrow[r, mapsto] & {[A]} \arrow[r, mapsto] & {[A]} \arrow[r, mapsto] & {[A]} \arrow[r, mapsto] & {[A]} \\
 & {[B]} \arrow[ur, mapsto] & {[B]} & {[C]} \arrow[r, mapsto] & {[C]} \arrow[ur, mapsto] & {[D]} \arrow[r, mapsto] & {[D]} \arrow[ur, mapsto] & \\
 & & & & {[D]} \arrow[ur, mapsto] & & & \\
 & {[A]+[B]} & & & & & &
\end{tikzcd}$$

Another example

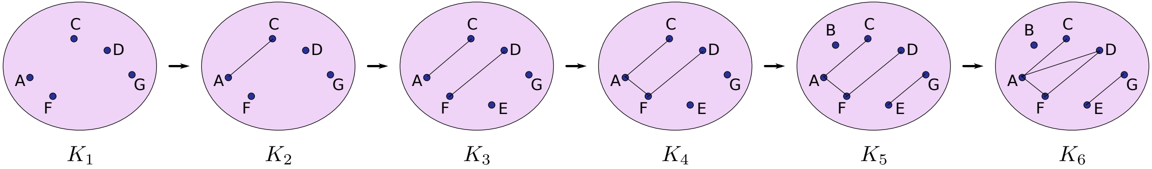


$$H_1(K_0) \xrightarrow{f_*} H_1(K_1) \xrightarrow{g_*} H_1(K_2) \xrightarrow{h_*} H_1(K_2)$$

$$[\alpha] \longmapsto [\alpha] \longmapsto 0$$

$$[\beta] \longmapsto [\gamma] \longmapsto [\delta]$$

Tryit: Figure out where the generators go



$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6)$$

$[A] \mapsto [A]$, $[C] \mapsto [A]$, $[D] \mapsto [D]$, $[F] \mapsto [F]$, $[G] \mapsto [G]$

$[A] \mapsto [A]$, $[D] \mapsto [D]$, $[F] \mapsto [D]$, $[G] \mapsto [G]$

$[A] \mapsto [A]$, $[D] \mapsto [A]$, $[E] \mapsto [E]$, $[G] \mapsto [G]$

$[A] \mapsto [A]$, $[E] \mapsto [E]$, $[G] \mapsto [E]$

$[A] \mapsto [A]$, $[B] \mapsto [B]$, $[E] \mapsto [E]$

The pth persistent homology groups

$$H_p(K_0) \to H_p(K_1) \to H_p(K_2) \to \cdots \to H_p(K_i) \to \cdots \to H_p(K_j) \to \cdots \to H_p(K_n)$$

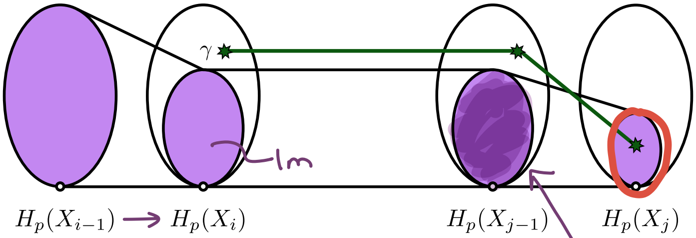
The pth persistent homology groups are the images induced by inclusion:

$$H_p^{i,j} = \mathrm{Im}(H_p(K_i) \to H_p(K_j)).$$

The pth persistent Betti numbers are the ranks

$$\beta_p^{i,j} = \mathrm{rank} H_p^{i,j}$$

Birth and Death



Birth

[α]

A class $\gamma \in H_p(K_i)$ is born at K_i if it is not in $H_p^{i-1,i}$

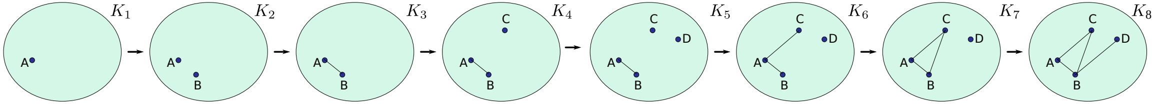
Death

That class γ dies entering K_j if merges with an older class.[a] Specifically, if $h_p^{i,j-1}(\gamma) \notin H_p^{i-1,j-1}$ but $h_p^{i,j}(\gamma) \in H_p^{i-1,j}$.

[a]Warning: The book's definition is different

When are generators born and die?

$T: V \to W \cong \mathrm{im}(T)$



$i-1=3$ $\quad$ $i=4$ $\quad$ $i-1=4$ $\quad$ $i=5$ $\quad$ $j=6$ $\quad$ $j=8$

$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

$[A] \mapsto [A] \mapsto [A] \mapsto [A] \mapsto [A] \mapsto [A] \mapsto [A] \mapsto [A]$

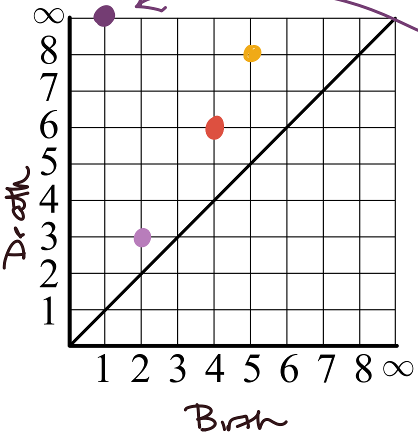
$[B] \mapsto [A]$

$[C] \mapsto [C] \mapsto [A]$

$[D] \mapsto [D] \mapsto [D] \mapsto [A]$

Elder rule and persistence diagram

When two classes merge, the younger is the one that "dies."

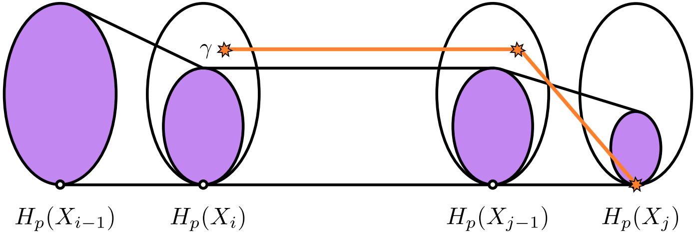


[A]
Born at 1
Never dies, lives forever
infinite class

Section 3

Dey Wang version of persistence classes

Birth and Death: Book version



Birth

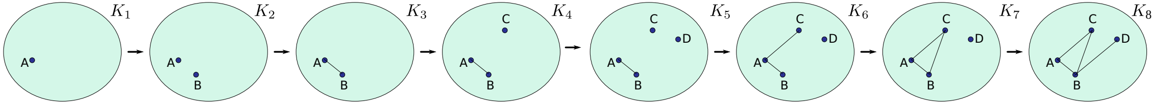
A class $\gamma \in H_p(K_i)$ is born at K_i if it is not in $H_p^{i-1,i}$

Death

A class γ dies entering K_j if $\gamma \in H_p(X_{j-1})$ is not trivial, but $h_p^{j-1,j}(\gamma) = 0$.[a]

[a]Warning: In this version, not all classes die!

Lets check where (lots of but maybe not all) elements go



$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

$0 \mapsto 0 \mapsto 0 \mapsto 0 \mapsto 0 \mapsto 0$

$[A] \mapsto [A] \mapsto [A] \mapsto [A] \mapsto [A] \mapsto [A]$

$[B] \mapsto [A]$

$[A]+[B] \mapsto 0$

never dies

$[C] \mapsto [C]$

$[A]+[C] \mapsto [A]+[C] \mapsto 0$

$[C] \mapsto [A]$

$[D] \mapsto [D]$

$[A]+[D] \mapsto [A]+[D]$

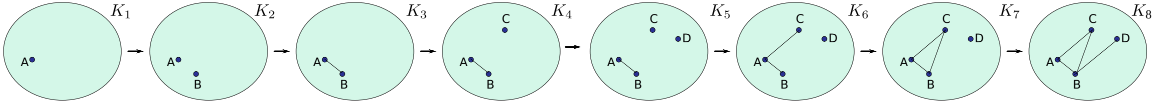
$[C]+[D] \mapsto [A]+[D]$

$[A]+[C]+[D] \mapsto [D]$

$[D]$

$[A]+[D]$

Choosing a different basis



$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

$[A]$ $[A]$ $[A]$ $[A]$ $[A]$ $[A]$ $[A]$ $[A]$

$[A]+[B] \longmapsto 0$

$[A]+[C] \longmapsto 0$

$[A]+[D] \longmapsto 0$

$i_1 = 1 \quad c_1 = [A]$

$i_2 = 4 \quad c_4 = [C]$

$\alpha = [A]+[C]$

Book definition: which birth time pairs to the death time?

maps to 0

Let $[c]$ be a p-th homology class that dies entering X_j. Then, it is born at X_i if and only if there exists a sequence

$$i_1 \leq i_2 \leq \cdots \leq i_k = i$$

for some $k \geq 1$ so that

- There is a c_{i_ℓ} where $[c_{i_\ell}]$ is born at X_{i_ℓ} for every $\ell \in \{1, \cdots, k\}$.
- $[\overset{\alpha}{\cancel{c}}] = h_p^{i_1,j-1}([c_{i_1}]) + \cdots + h_p^{i_k,j-1}([c_{i_k}])$
- $i_k = i$ is the smallest possible value among any sequences have the above two properties.

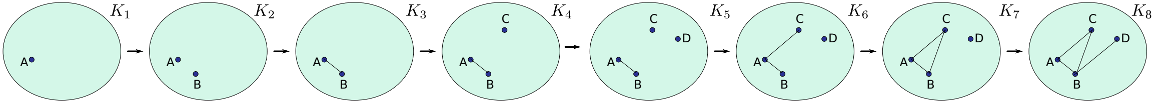
Counting classes

Persistence Module

$$0 \to H_p(K_1) \to H_p(K_2) \to H_p(K_3) \to \cdots \to H_p(K_n) \to 0$$

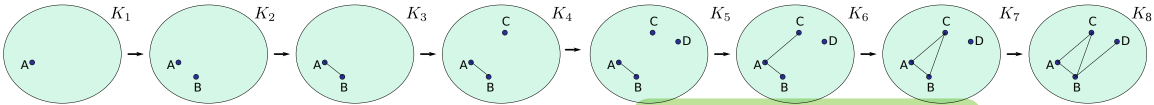
- Attach 0 vector space at the end
- Associate $n+1$ to $a_{n+1} = \infty$
- $\beta_p^{i,j} = \mathrm{rank} H_p^{i,j} = \mathrm{rank}(\mathrm{Im}(H_p(K_i) \to H_p(K_j)))$
  counts classes born **at or before** i and dying **after** j

Same example again



$$
\begin{array}{ccccccccccccccc}
H_0(K_1) & \longrightarrow & H_0(K_2) & \longrightarrow & H_0(K_3) & \longrightarrow & H_0(K_4) & \longrightarrow & H_0(K_5) & \longrightarrow & H_0(K_6) & \longrightarrow & H_0(K_7) & \longrightarrow & H_0(K_8) \\
[A] & \longmapsto & [A] & \longmapsto & [A] & \longmapsto & [A] & \longmapsto & [A] & \longmapsto & [A] & \longmapsto & [A] & \longmapsto & [A] \\
 & & [A+B] & \longmapsto & 0 & & [A+C] & \mapsto & [A+C] & \longmapsto & 0 & & & & \\
 & & & & & & & & [A+D] & \mapsto & [A+D] & \mapsto & [A+D] & \longmapsto & 0
\end{array}
$$

TRYIT: Example



j \ i	1	2	3	4	5	6	7	8	∞
∞									
8	1	1	1	1	1	1			
7	1	2	1	1	2	2			
6	1	1	1	1	2	2			
5	1	1	1	2	3	X			
4	1	2	1	2	X	X			
3	1	1	1	X	X	X			
2	1	2	X	X	X	X			
1	1	X	X	X	X	X			

Determine $\beta_0^{i,j}$ for all pairs $i \leq j$ for this example

Persistence pairing function

How to get the number of classes born **at** i and dying **at** j???

For $0 < i < j \leq n+1$, define

Born $\leq i$
Die $\geq j$

Born $\leq i$
Die $\geq j+1$

$$\mu_p^{i,j} = \begin{cases} (\beta_p^{i,j-1} - \beta_p^{i,j}) - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j}) & i < j < n+1 \\ \beta_p^{i,j} - \beta_p^{i-1,j} & i < j = n+1 \end{cases}$$

Born $\leq i-1$
Die $\geq j$

Born $\leq i-1$
Die $\geq j+1$

Born $\leq i$
Die $= j$

Born $\leq i-1$
Die $= j$

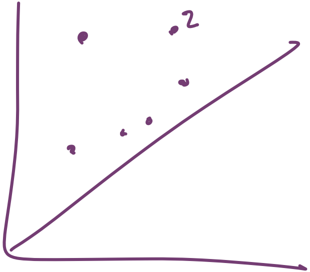
Persistence of a class

For $\mu_p^{i,j} \neq 0$, the persistence $Pers([c])$ of a class $[c]$ that is born at X_i and dies at X_j is defined as $Pers([c]) = \underline{a_j - a_i}$. When $j = n + 1$ with $a_{n+1} = \infty$, $Pers([c]) = \infty$.

The lifetime

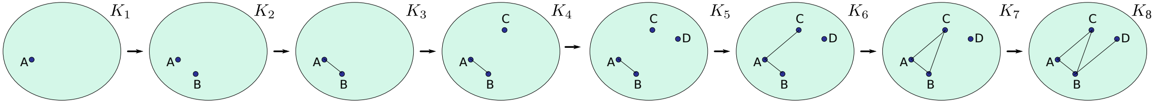
Persistence diagram

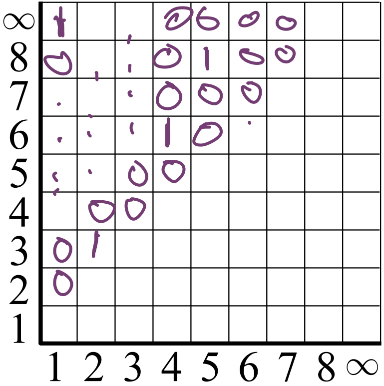
The persistence diagram $Dgm_p(\mathcal{F})$ (also written $Dgm_p(f)$) of a filtration induced by a function f is obtained by drawing a point (a_i, a_j) with non-zero multiplicity $\mu_p^{i,j}$ $(i < j)$, on the extended plane where the points on the diagonal $\Delta = \{(x, x) \in \mathbb{R}^2\}$ are added with infinite multiplicity.

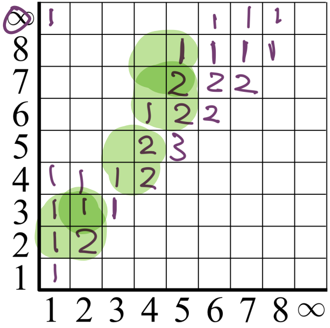


multiple things can be born and die @ same time

TRYIT: Example





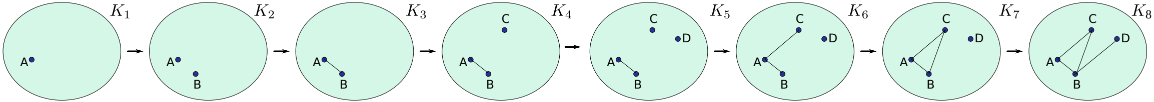


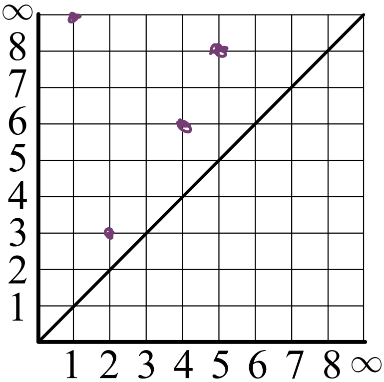
j for this example

$^{1,j-1} - \beta_p^{i-1,j})$ $\quad i < j < n+1$

$i < j = n+1$

TRYIT: Persistence diagram for this example



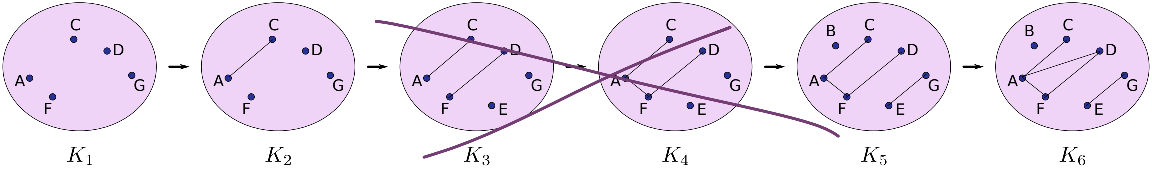


Plot the 0-dimensional persistence diagram

Homework

Jannik

Determine the $\beta_0^{i,j}$ table, the $\mu_0^{i,j}$ table, and the 0-dimensional persistence diagram for the filtration below.



Gn version