

Multiparameter Persistence and RIVET

Lecture 23 - CMSE 890

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Tues, Dec 2, 2025

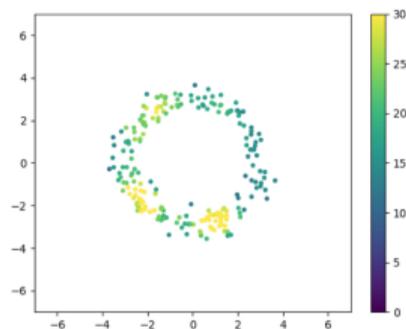
Goals for today

- Multiparamter persistence survey
 - ▶ Botnan, M. B., & Lesnick, M. (2022, January 1). An introduction to multiparameter persistence. arXiv:2203.14289
- RIVET
 - ▶ <https://rivet.readthedocs.io>
- Crocker plots:
 - ▶ Topaz, Ziegelmeier, & Halverson, Topological data analysis of biological aggregation models. PLoS ONE 2015. 10.1371/journal.pone.0126383

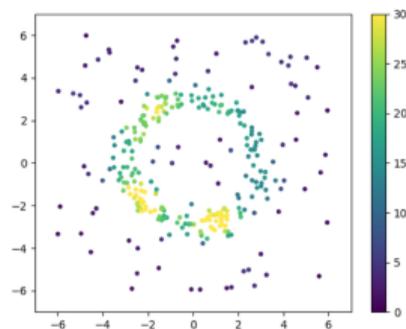
Section 1

Mathematical preliminaries

The problem



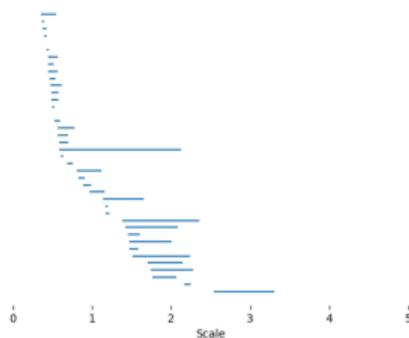
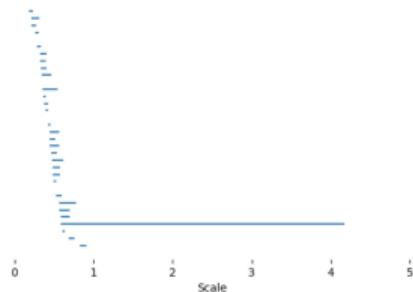
(a)



(b)

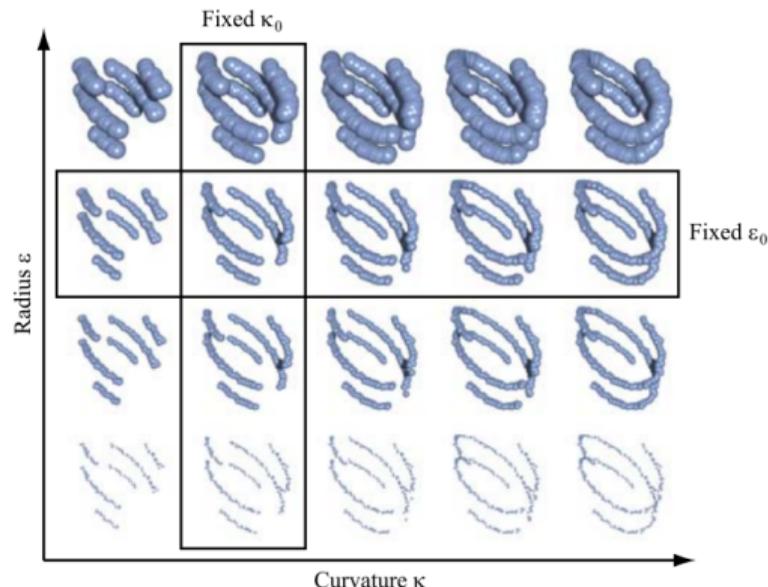
- Colored by local density
- Persistence is not stable to ambient noise

Figure: Botnan and Lesnick 2022 survey

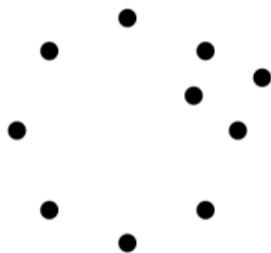


Bifiltration

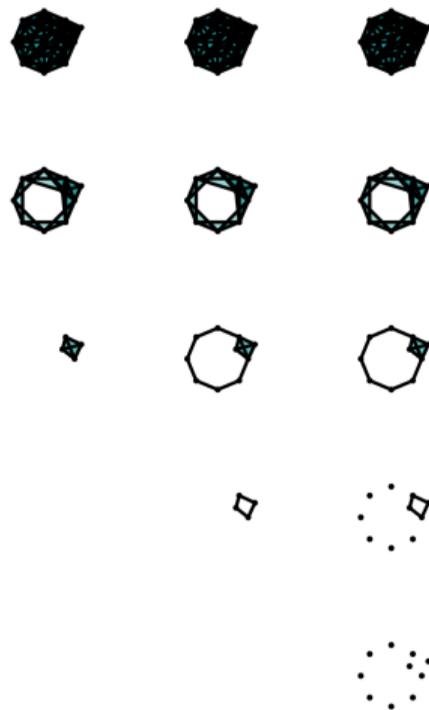
A bifiltration F is a collection of finite simplicial complexes indexed by \mathbb{Z}^2 such that $F_{(a_1, a_2)} \subseteq F_{(b_1, b_2)}$ whenever $a_1 \leq b_1$ and $a_2 \leq b_2$.



Bfiltration from function



- x-axis: filter by density f_x
- y-axis: filter by scale f_y
- $K_{a,b} = \{\sigma \in K \mid f_x(\sigma) \leq a, f_y(\sigma) \leq b\}$



Function options implemented in RIVET

- **Ball density function:** (r fixed parameter; C normalization constant so that $\sum \gamma(x) = 1$)

$$\gamma(x) = C \cdot (\# \text{ points in } P \text{ within distance } r \text{ of } x)$$

- **Gaussian density function** ($\sigma > 0$ parameter, C normalization constant)

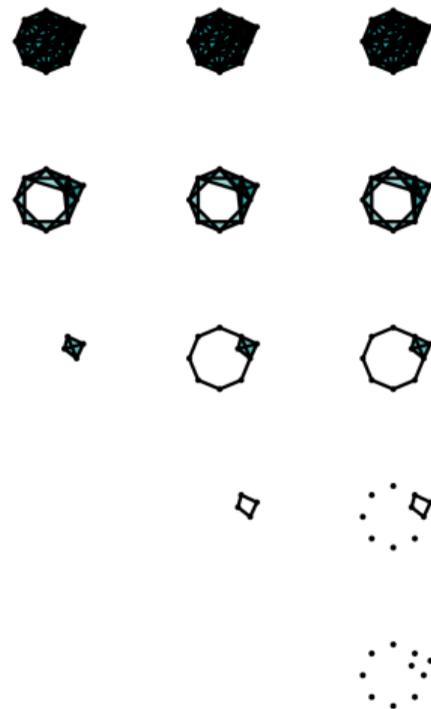
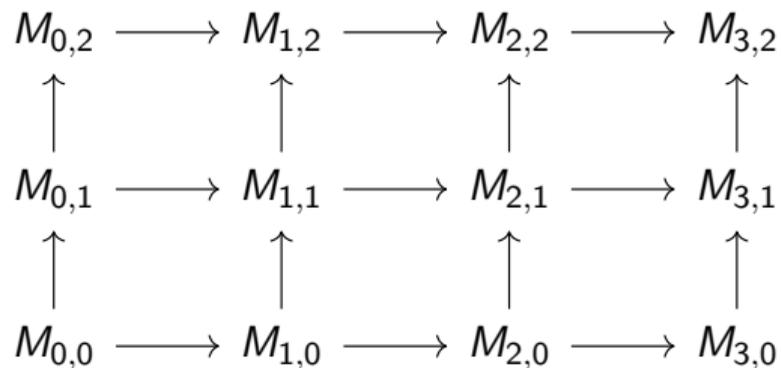
$$\gamma(x) = C \sum_{y \in P} \exp\left(\frac{-d(x, y)^2}{2\sigma}\right)$$

- **Eccentricity function** ($q \in [1, \infty)$)

$$\gamma(x) = \left(\frac{\sum_{y \in P} d(x, y)^q}{|P|}\right)^{\frac{1}{q}}$$

Bipersistence module

M is a collection of vector spaces $\{M_a\}_{a \in \mathbb{Z}^2}$ with maps $\{M_{a,b} : M_a \rightarrow M_b\}_{a \leq b}$



No free lunch

Theorem (Carlsson & Zomorodian 2009 (paraphrased))

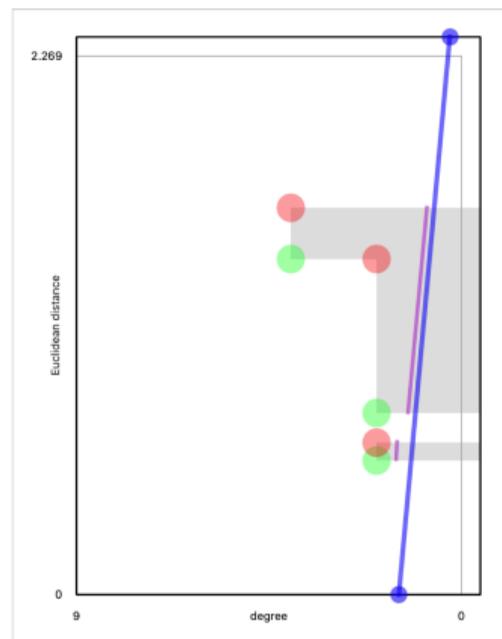
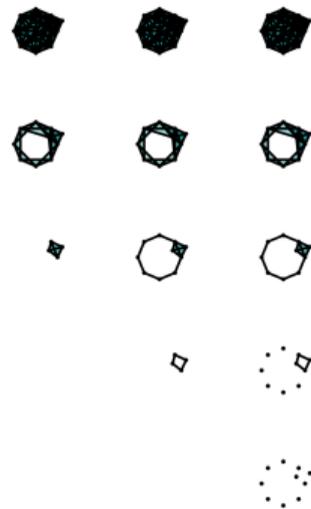
The algebraic classification of indecomposables of multiparameter persistence modules contain both discrete and continuous portions, meaning that no persistence-diagram-like representation is available.

The Theory of Multidimensional Persistence. Carlsson & Zomorodian 2009

Invariants of a bipersistence module

The Hilbert function

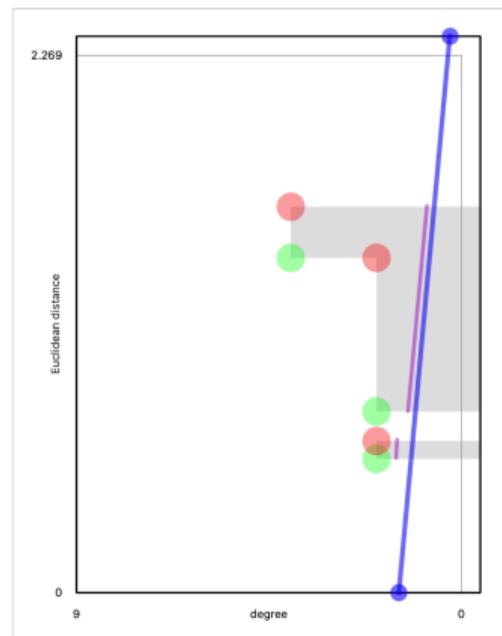
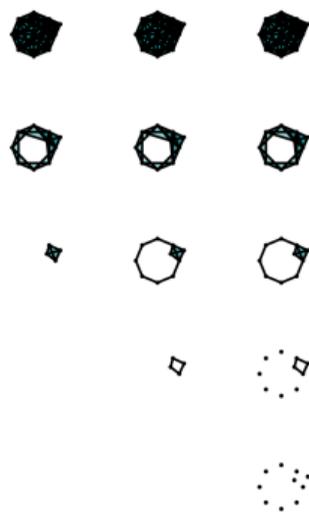
The Hilbert function $Hilb^M : \mathbb{R}^2 \rightarrow \mathbb{N}$ counts the dimension of the vector space at each location, $(a, b) \mapsto \dim(M_{a,b})$.



Invariants of a bipersistence module

Bigraded Betti numbers

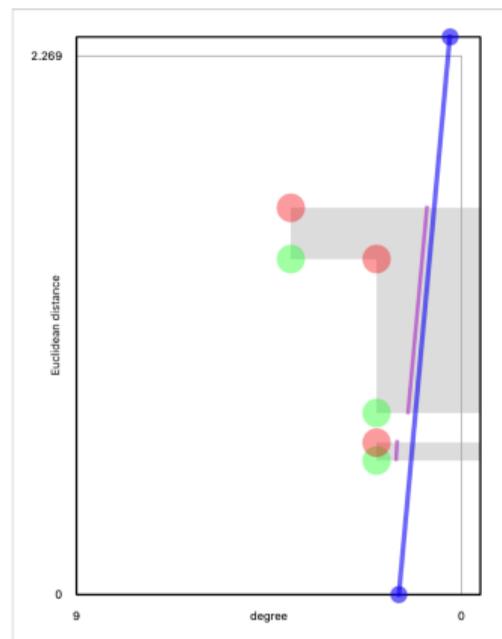
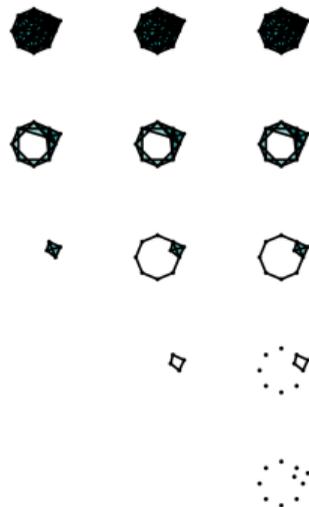
- The bigraded Betti numbers ζ_i^M . These are functions $\mathbb{R}^2 \rightarrow \mathbb{N}$ that, respectively, count the number of births, deaths, and “relations amongst deaths” at each bigrade.
- *Warning: Not the same as the Betti numbers of a space!*
- Related to minimal free resolution of M
- ζ_0 : Red dots; ζ_1 : Green dots; ζ_2 : Yellow dots



Invariants of a bipersistence module

Fibered Barcode

- Restrict to a line L with positive slope
- Draw the barcode/persistence diagram for that line



Example: HIV Data

Option 1

- 0-dimensional persistence
- Metric for Rips: Hamming distance

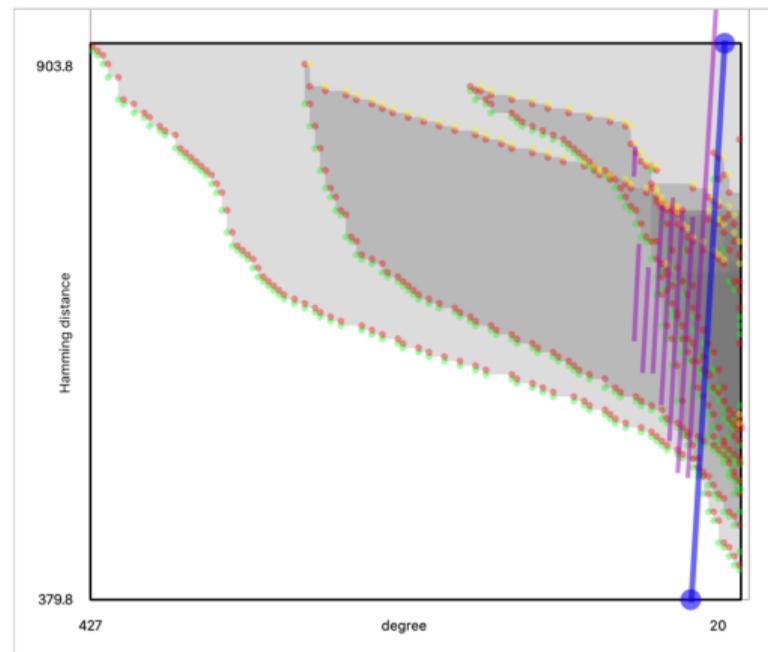


Figure: Botnan and Lesnick 2022 survey; Data: Chan et al. Topology of viral evolution. PNAS 2013.

Example: HIV Data

Option 2

- 0-dimensional persistence
- Metric for Rips: Hamming distance

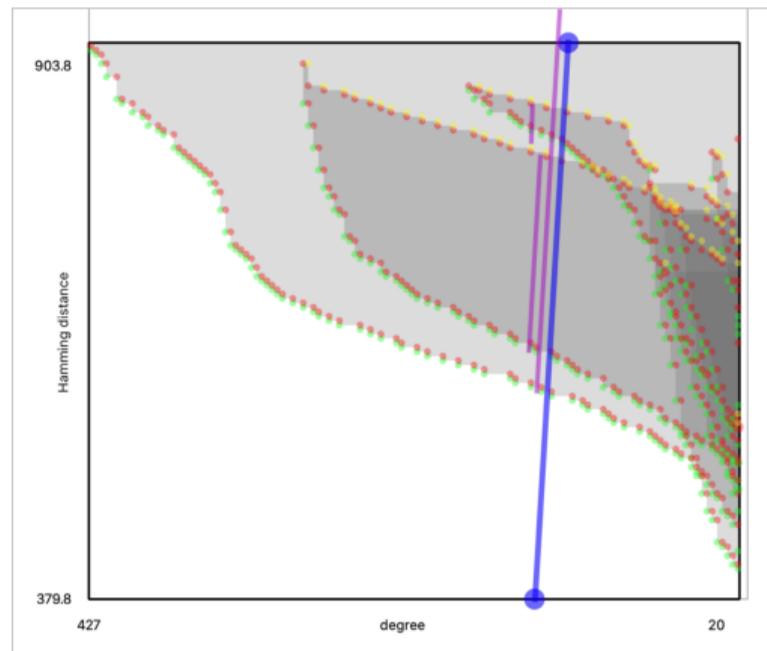


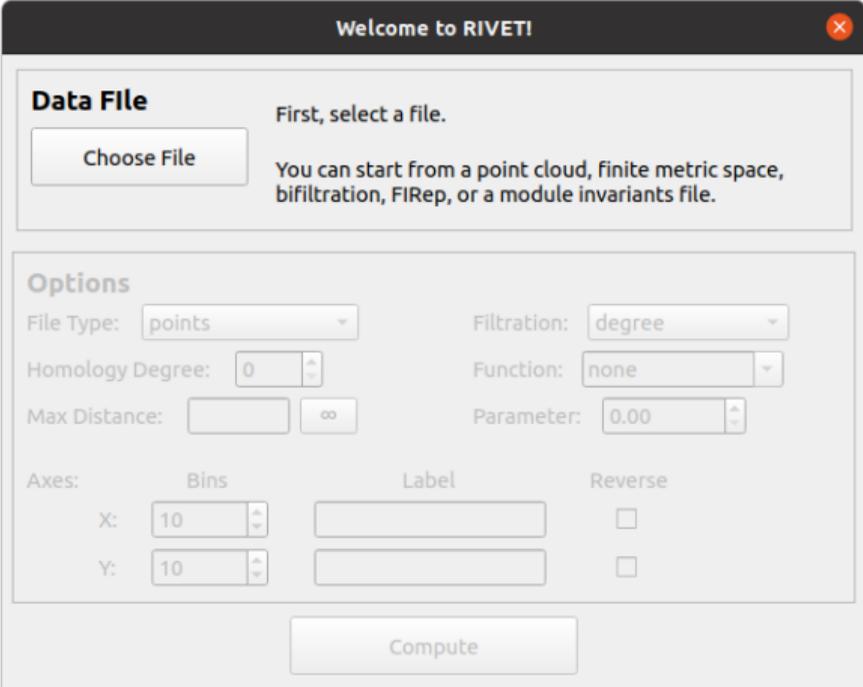
Figure: Botnan and Lesnick 2022 survey; Data: Chan et al. Topology of viral evolution. PNAS 2013.

Section 2

Using RIVET

Running the GUI

Run rivet_GUI



The screenshot shows a window titled "Welcome to RIVETI" with a close button in the top right corner. The window is divided into two main sections: "Data File" and "Options".

Data File

First, select a file.

You can start from a point cloud, finite metric space, bifiltration, FIRep, or a module invariants file.

Options

File Type: Filtration:

Homology Degree: Function:

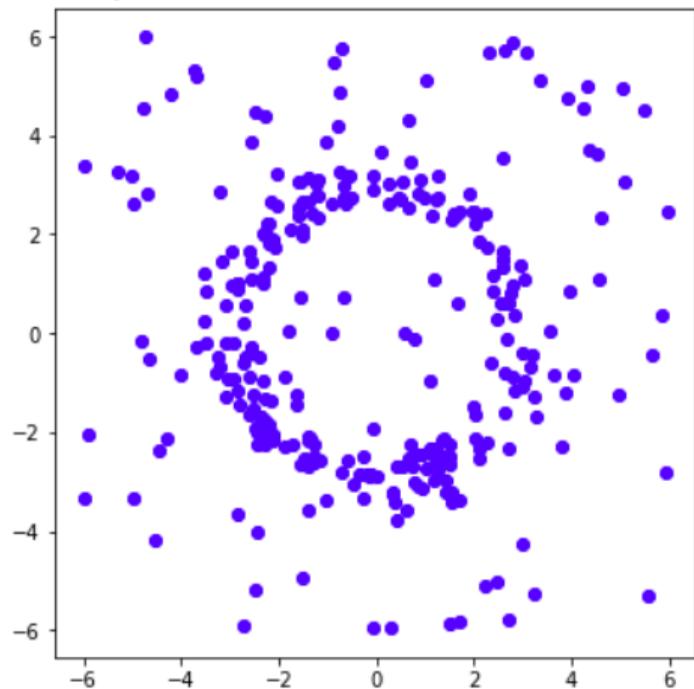
Max Distance: Parameter:

Axes: Bins Label Reverse

Axis	Bins	Label	Reverse
X:	<input type="text" value="10"/>	<input type="text"/>	<input type="checkbox"/>
Y:	<input type="text" value="10"/>	<input type="text"/>	<input type="checkbox"/>

Test data

Example included in install: `data/Test_Point_Clouds/circle_300pts_nofunction.csv`



Inputs for this example

Welcome to RIVET! ✕

Data File

Selected file: circle_300pts_nofunction.csv

This file appears to contain point-cloud data.

Options

File Type:

Filtration:

Homology Degree:

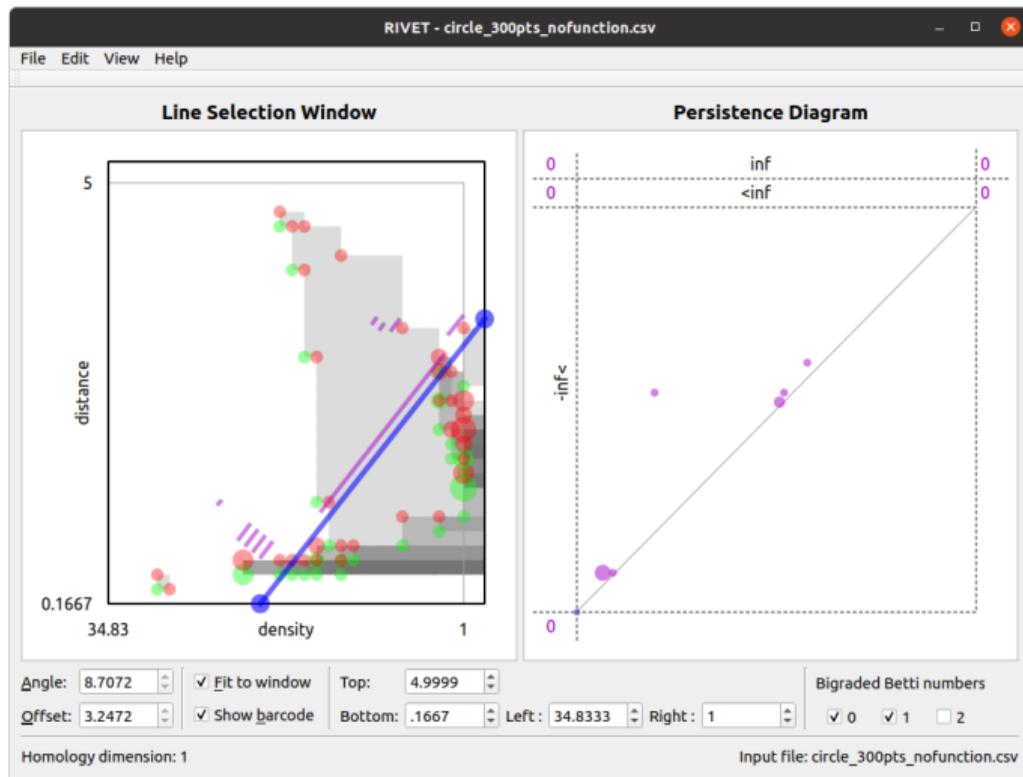
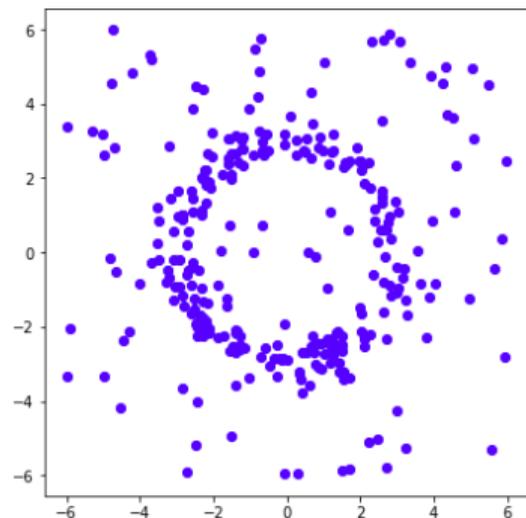
Function:

Max Distance:

Radius:

Axes:	Bins	Label	Reverse
X:	<input type="text" value="30"/>	<input type="text" value="density"/>	<input checked="" type="checkbox"/>
Y:	<input type="text" value="30"/>	<input type="text" value="distance"/>	<input type="checkbox"/>

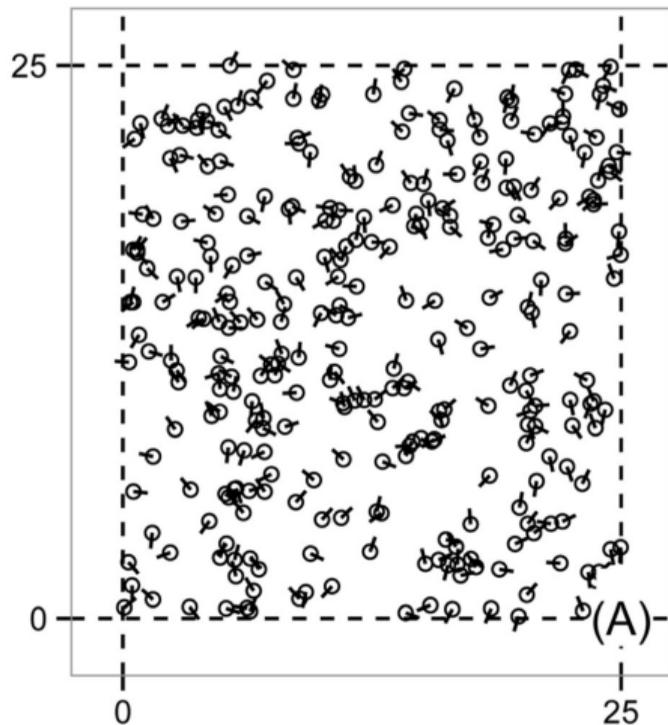
The interactive console



Section 3

Crocker Plots

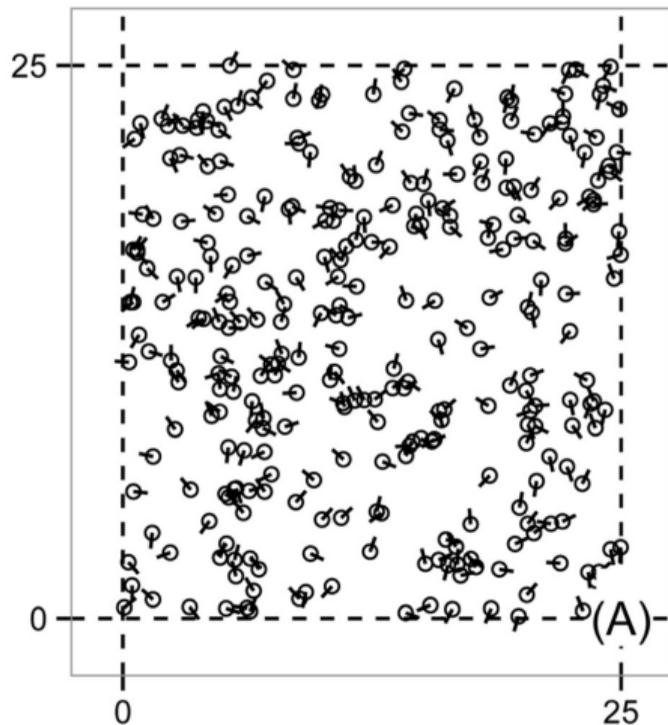
Data: Flocking



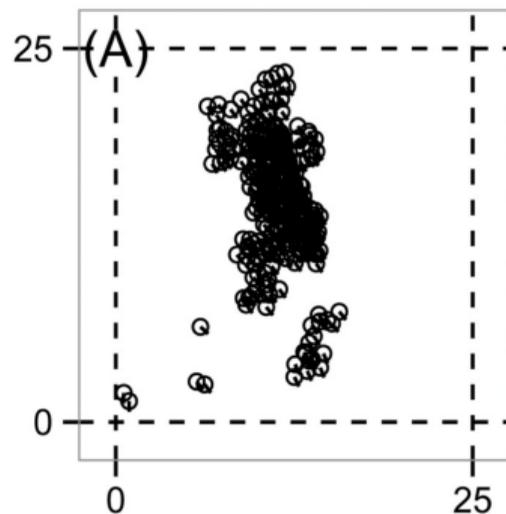
Left: Wikipedia; Right: Topaz et al 2015.

Vicsek Model

- Discrete time, continuous space
- Motion of interacting point particles in a square with periodic boundary conditions
- Updates:
 - ▶ θ : Angle of motion updated to average angle of neighbors + noise
 - ▶ Position updated by moving in direction of θ with fixed speed
 - ▶ Parameters: number of particles, density, noise level, interaction radius, speed

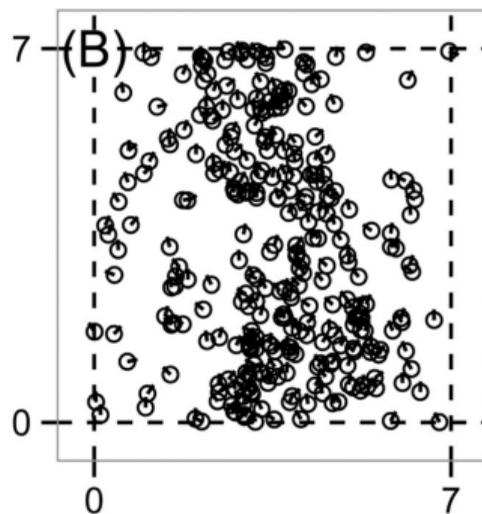


Examples



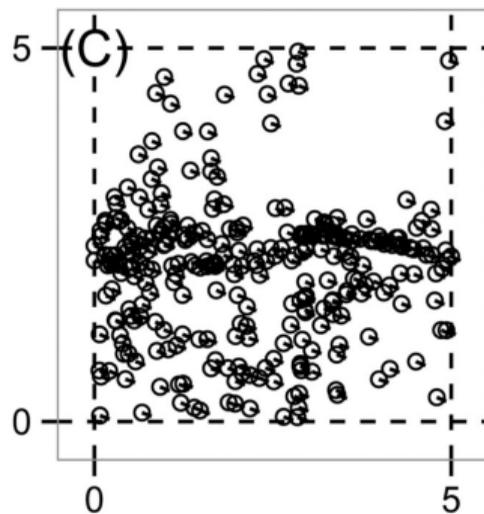
Groups moving in different directions

$$\ell = 25, \eta = 0.1$$



Random movement with some correlation

$$\ell = 7, \eta = 2$$

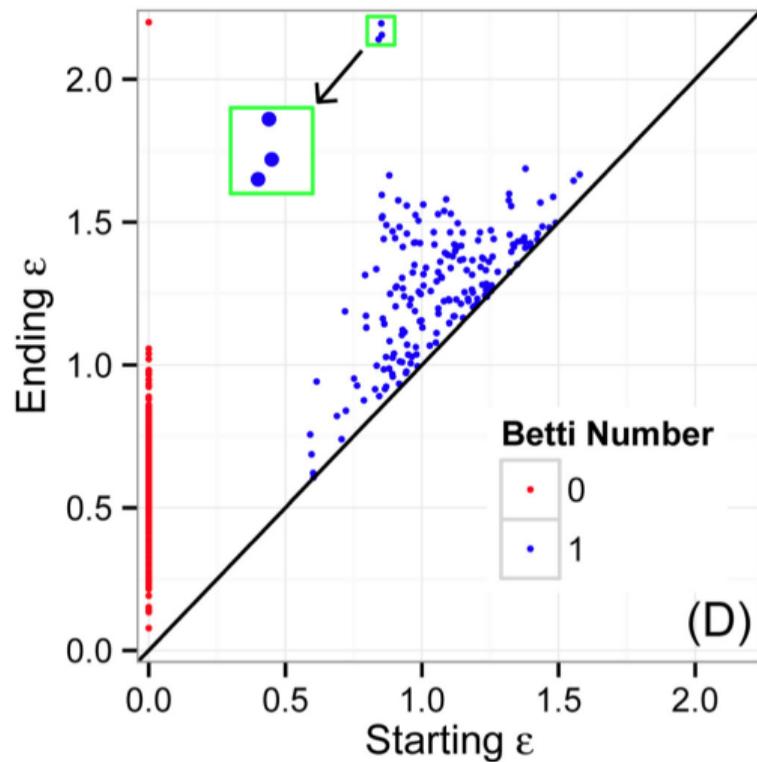
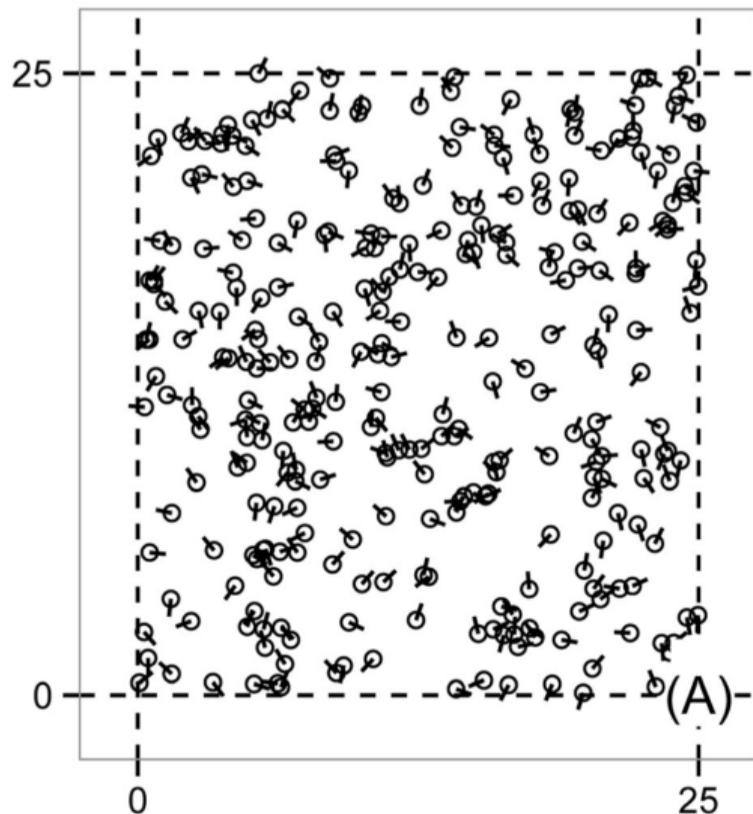


Highly polarized movement

$$\ell = 5, \eta = 0.1$$

Boxsize ℓ , Noise η

Persistence at each time point

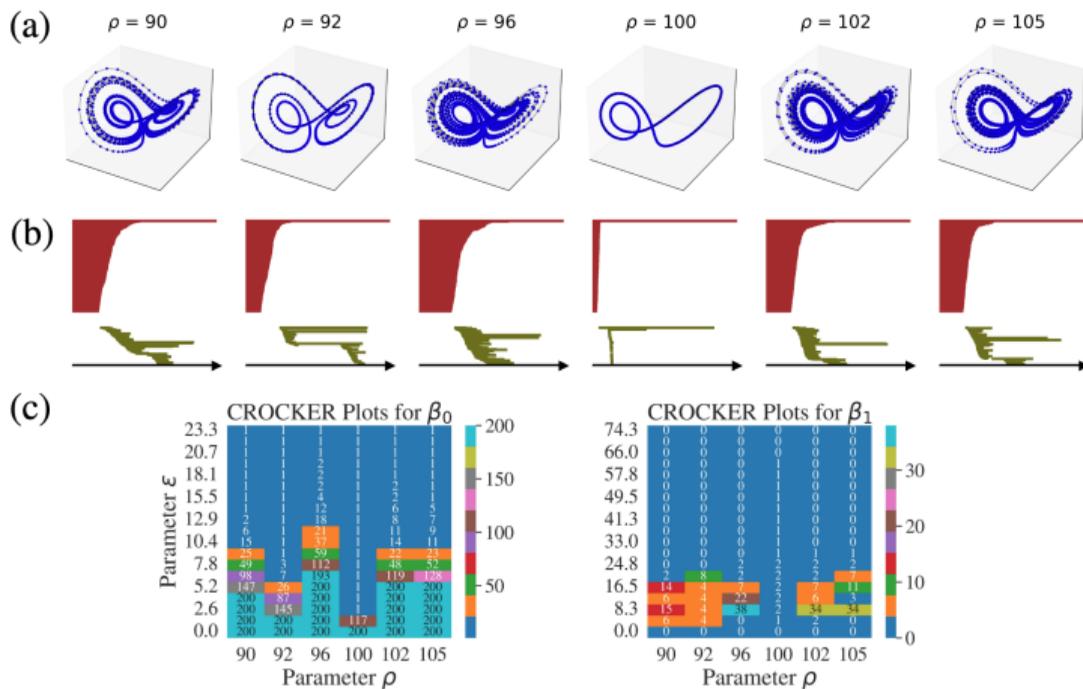


Dynamic Metric Space

Definition 7.1 ([58]) A *dynamic metric space* is a pair $\gamma_X = (X, d_X(\cdot))$ where X is a nonempty finite set and $d_X : \mathbf{R} \times X \times X \rightarrow \mathbf{R}_+$ satisfies:

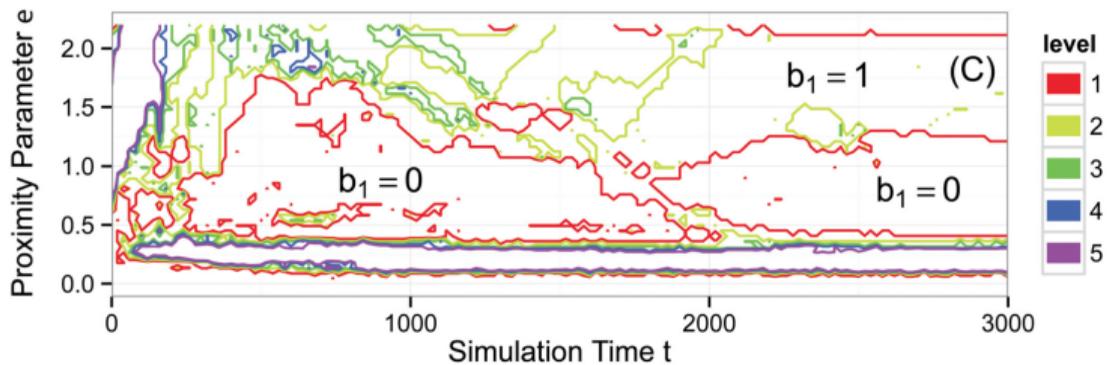
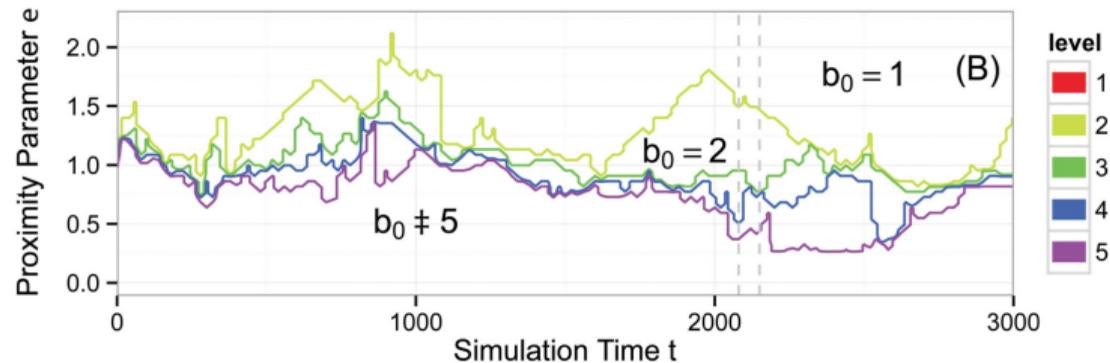
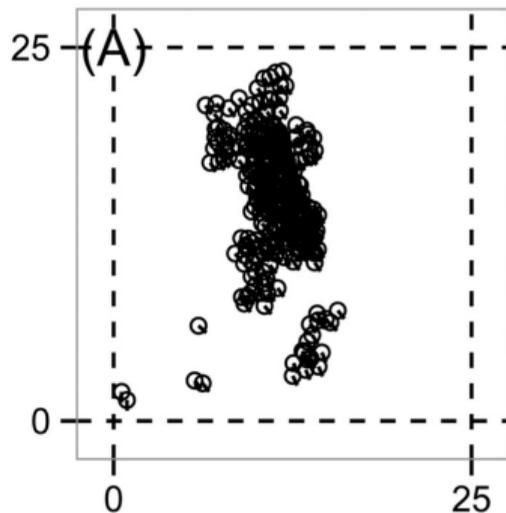
- (i) For every $t \in \mathbf{R}$, $\gamma_X(t) = (X, d_X(t))$ is a pseudo-metric space.
- (ii) For any $x, x' \in X$ with $x \neq x'$ the function $d_X(\cdot)(x, x') : \mathbf{R} \rightarrow \mathbf{R}_+$ is not identically zero.
- (iii) For fixed $x, x' \in X$, $d_X(\cdot)(x, x') : \mathbf{R} \rightarrow \mathbf{R}_+$ is continuous.

Crocker Plot



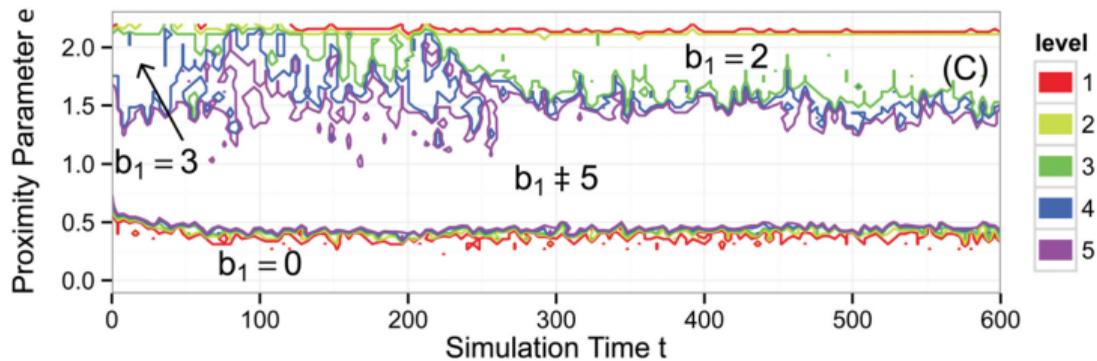
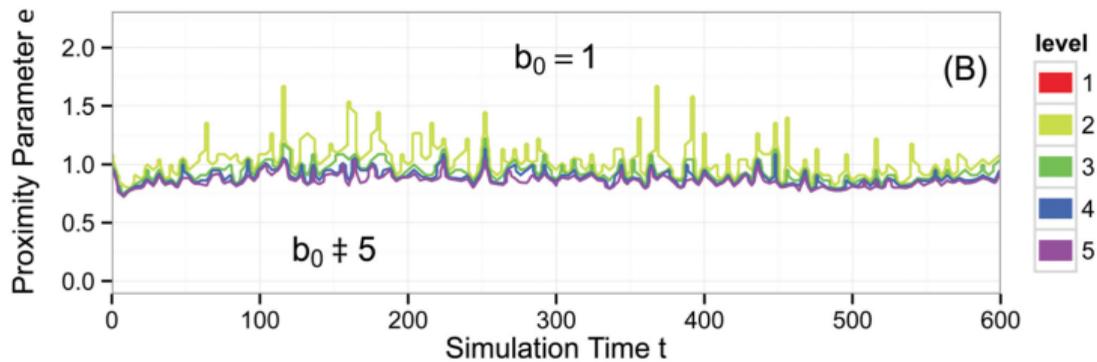
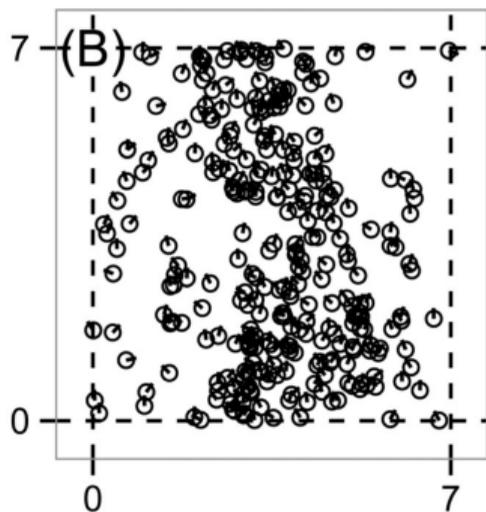
Crocker Plots

$N = 300$, $\ell = 25$, $\eta = 0.1$



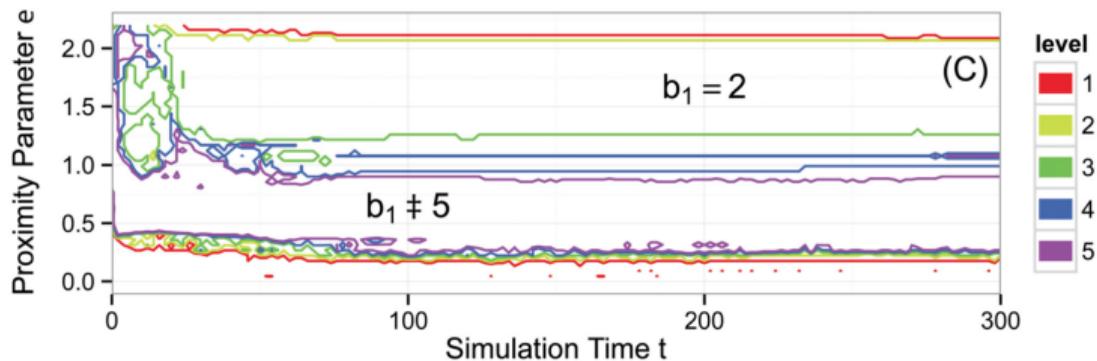
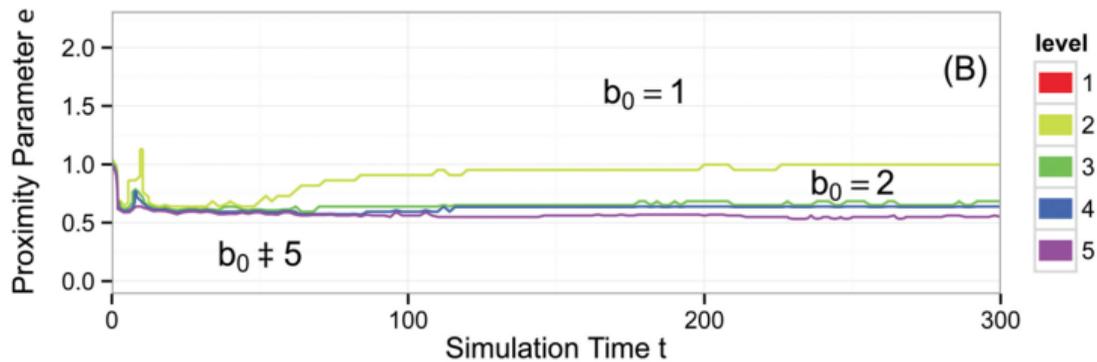
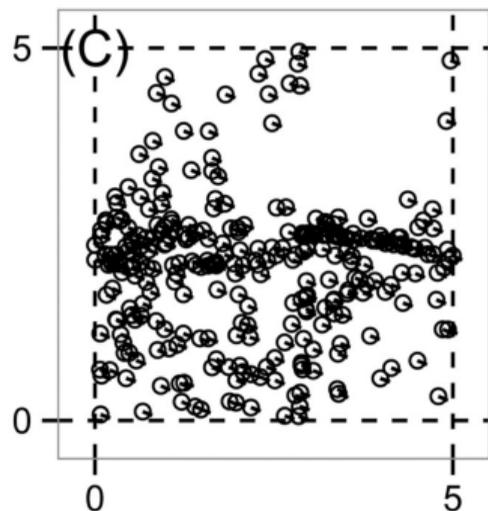
Crocker Plots

$N = 300$, $\ell = 7$, $\eta = 2$



Crocker Plots

$N = 300$, $\ell = 5$, $\eta = 0.1$



Next time

- Time series analysis