The Persistent Homology Transform and Monodromy
Lecture 21 - CMSE 890

Prof. Elizabeth Munch

Michigan State University
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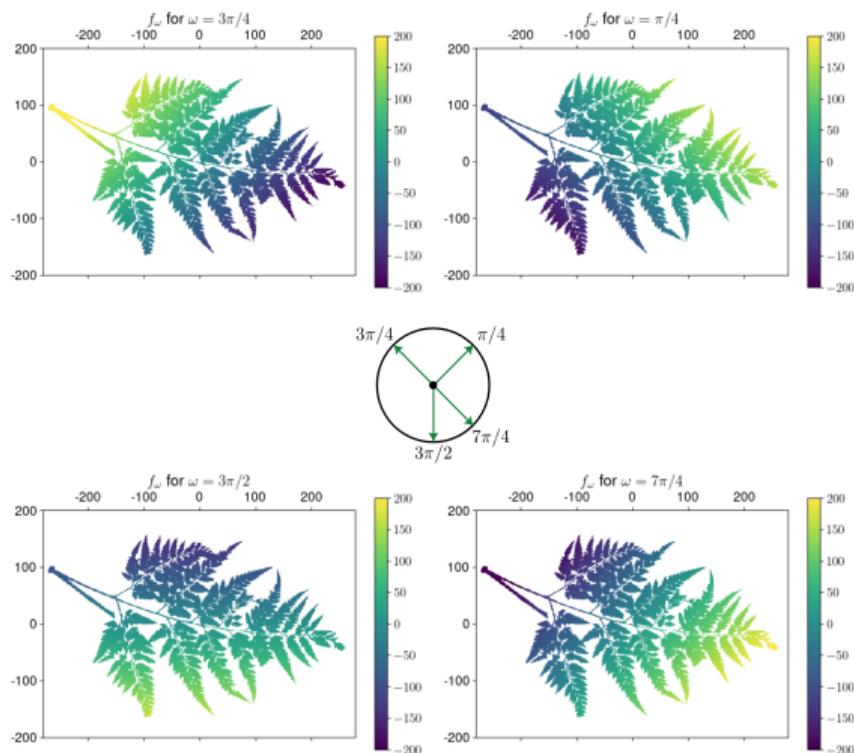Dept of Computational Mathematics, Science & Engineering

Thurs, Nov 20, 2025

Plan

- The Persistent Homology Transform (PHT)
- Monodromy

Section 1

From Last Time: PHT

Direction induces a function

- Fix a subset of $\mathbb{R}^d$, A
- Fix a direction $\omega \in \mathbb{S}^{d-1}$
- $f_\omega : A \to \mathbb{R}$, $f_\omega(x) = \langle x, \omega \rangle$

Function induces an algebraic representation

- χ_ω = Euler characteristic curve of f_ω
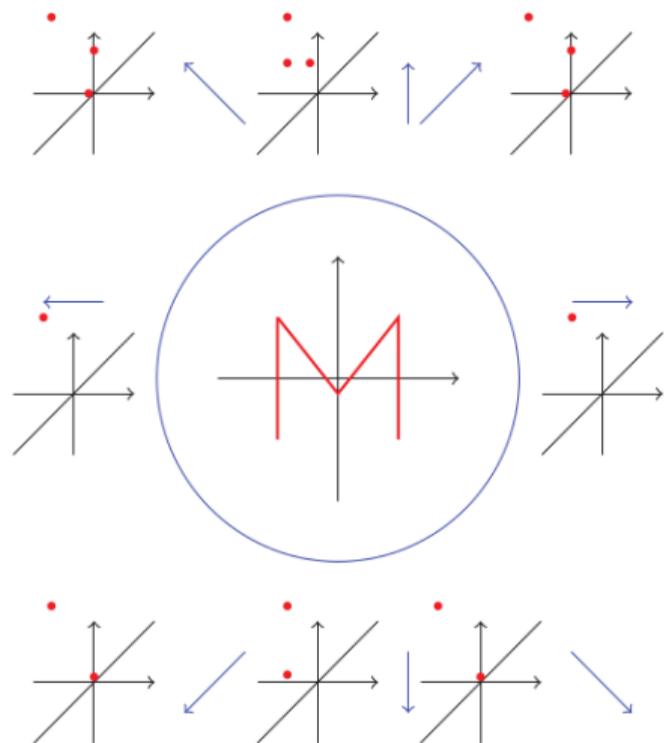- $Pers_{k,\omega}$ = k-dimensional persistence diagram of f_ω

Figure from: Turner, Mukherjee, Boyer. Persistent homology transform for modeling shapes and surfaces.

The full transform(s)

$$PHT(A) : \begin{aligned} \mathbb{S}^{d-1} &\longrightarrow \mathcal{D}gm^d \\ \omega &\longmapsto (Pers_{0,\omega}(A), \ldots, Pers_{d-1,\omega}(A)) \end{aligned}$$

$$ECT(A) : \begin{aligned} \mathbb{S}^{d-1} &\longrightarrow \text{Functions on } \mathbb{R} \\ \omega &\longmapsto \chi_\omega(A) \end{aligned}$$

Meta transforms

$\mathcal{M}_d$ = space of all finite simplicial complexes embedded in $\mathbb{R}^d$

$$PHT : \quad \mathcal{M}_d \quad \longrightarrow \quad \text{Functions from } \mathbb{S}^{d-1} \text{ to } \mathcal{D}gm^d$$
$$A \quad \longmapsto \quad \omega \mapsto (Pers_{0,\omega}(A), \ldots, Pers_{d-1,\omega}(A))$$

$$ECT : \quad \mathcal{M}_d \quad \longrightarrow \quad \text{Functions from } \mathbb{S}^{d-1} \text{ to (Functions on } \mathbb{R})$$
$$A \quad \longmapsto \quad \omega \mapsto \chi_\omega(A)$$

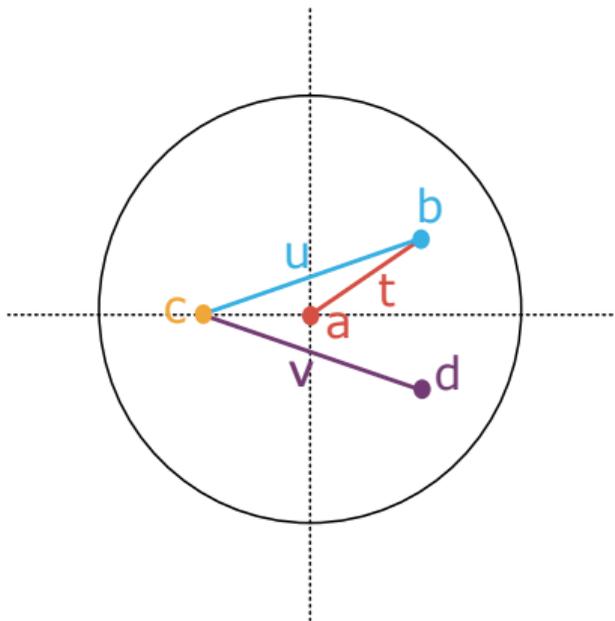# Injectivity theorem

Theorem (Turner, Mukherjee, Boyer)

The ECT and PHT are injective on the space $\mathcal{M}_d$
of finite simplicial complexes embedded in $\mathbb{R}^2$ or $\mathbb{R}^3$.
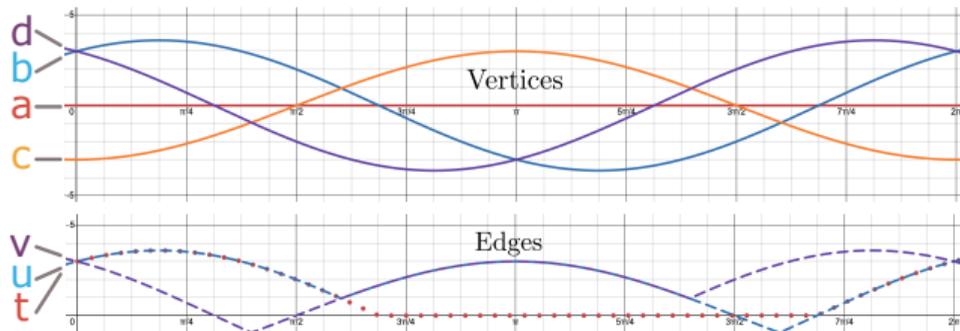
Generalizations by Ghrist et al; Curry et al

Section 2

A smaller example

Input space

$f(\cdot, \sigma)$ for each $\sigma \in K$

Vertices

Edges

https://www.desmos.com/calculator/ft1jbnlysm

Breaking up $\mathbb{S}^1$
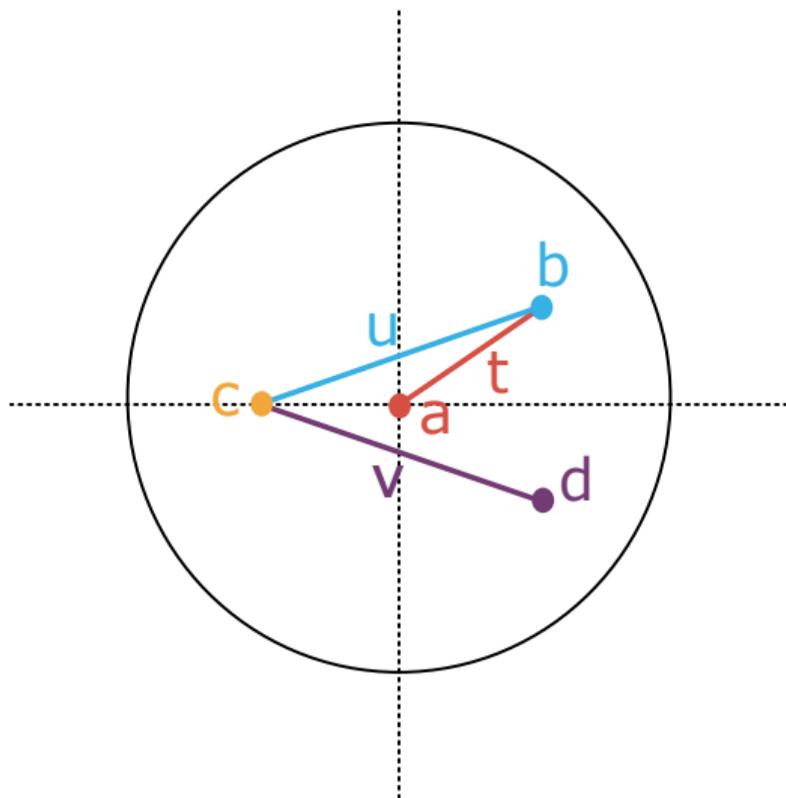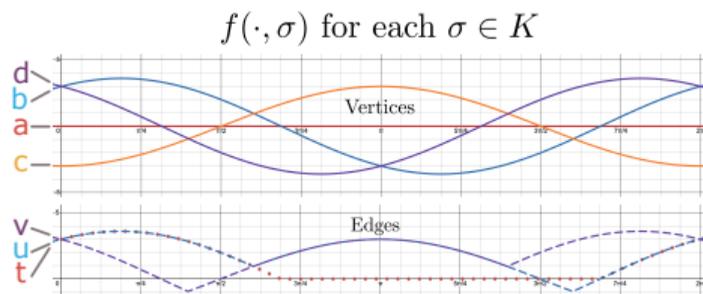
Order of vertices inside regions

$f(\cdot, \sigma)$ for each $\sigma \in K$

Vertices

Edges
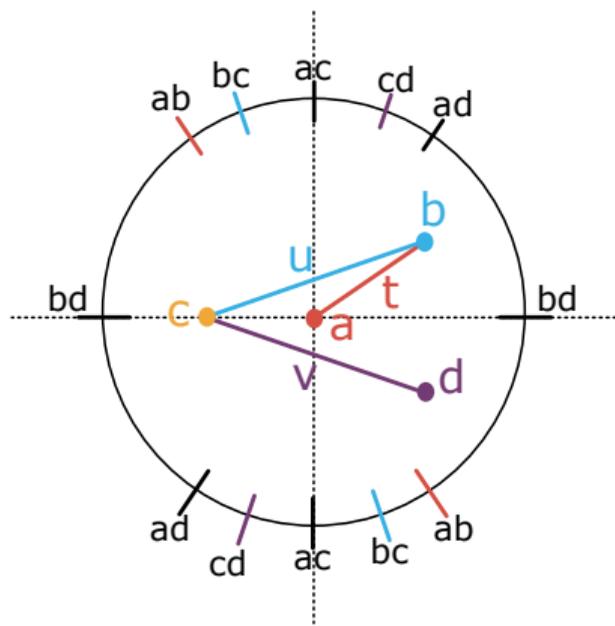
Persistence diagrams

Example for $[0, \sim \pi/4]$: https://www.desmos.com/calculator/xzqm2g98p6
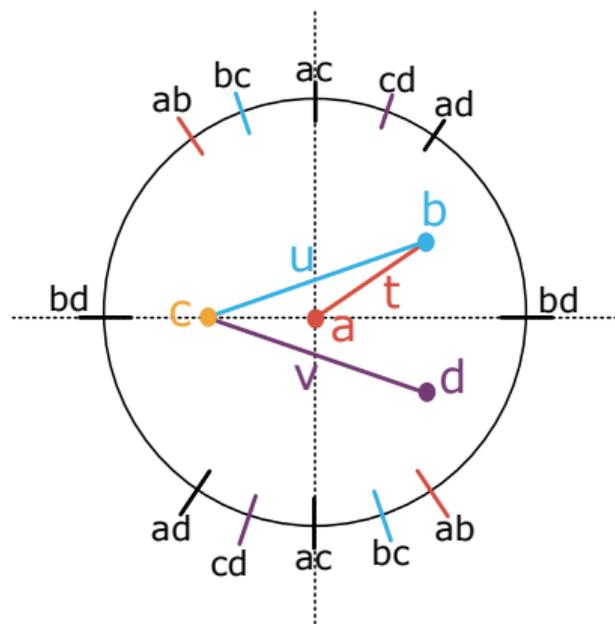
Section 3

Finitely Many Directions

The problem of needing all directions

$\mathcal{M}_d =$ space of all finite simplicial complexes embedded in $\mathbb{R}^d$

$$PHT : \begin{array}{ccc} \mathcal{M}_d & \longrightarrow & \text{Functions from } \mathbb{S}^{d-1} \text{ to } \mathcal{D}gm^d \\ A & \longmapsto & \omega \mapsto (Pers_{0,\omega}(A), \dots, Pers_{d-1,\omega}(A)) \end{array}$$

Controlling the complexes

Definition

Let $\mathcal{K}(d, \delta, k_\delta)$ be the set of embedded simplicial complexes K in $\mathbb{R}^d$ such that:

- Every vertex $x \in K$ is at least δ-observable.
  - $\exists$ a ball of directions $B(v_x, \delta) \subseteq \mathbb{S}^d$ such that the homology changes at $\langle v, x \rangle$ for all $v \in B(v_x, \delta)$.
- For all $v \in \mathbb{S}^{d-1}$, the number of $x \in X$ for which $h_w(x)$ is a homological critical value of h_w for some $w \in B(v, \delta)$ is bounded by k_δ.
  - k_δ is a global bound on the number of vertices that can change the homology for any ball $B(v, \delta)$.

Finitely many directions

Theorem (Curry, Mukherjee, Turner 2022)

Any shape in $\mathcal{K}(d, \delta, k_\delta)$ can be determined using the PHT using no more than

$$((d-1)k_\delta + 1)\left(1 + \frac{3}{\delta}\right)^d + O\left(\frac{d^{d+1}k_\delta^{2d}}{\delta^{2d(d-1)}}\right)$$

From: Curry, Mukherjee, Turner. How many directions determine a shape and other sufficiency results for two topological transforms. 2022.

Section 4

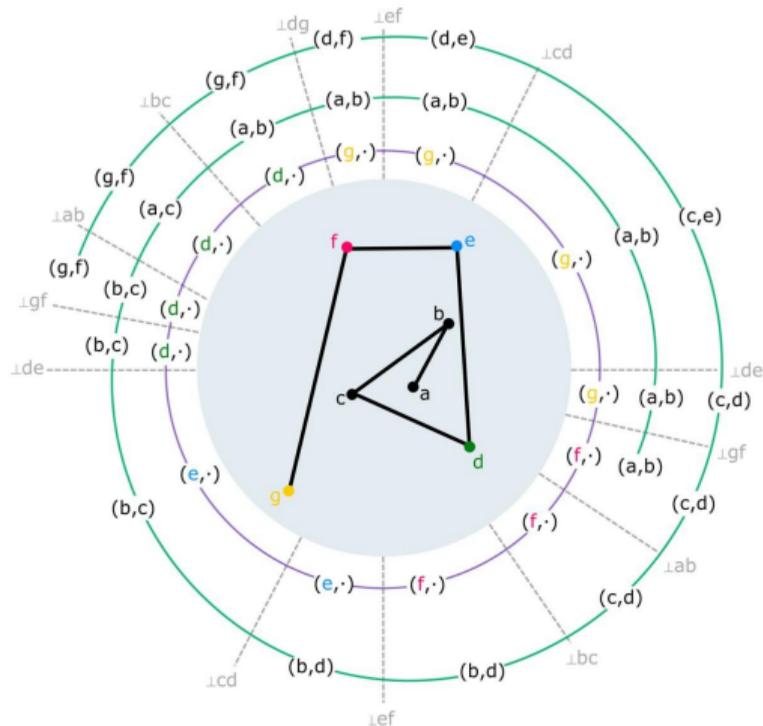## A bigger example
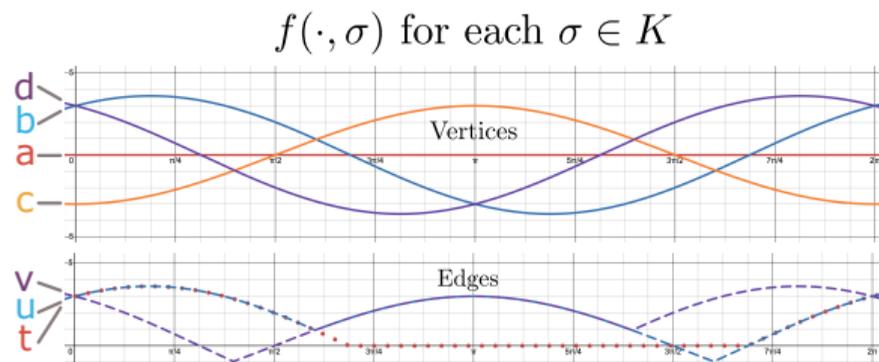
Monodromy Example

Figure from Arya et al., Decomposing the Persistent Homology Transform of Star Shaped Objects 2024.

Fibered filtration function

Definition

A **fibered filtration function** is a continuous function $f : K \times B \to \mathbb{R}$. For each $p \in B$, the function $f_p : K \to \mathbb{R}$ defined by $f_p(\sigma) = f(\sigma, p)$ is a filtration function on K.

$f(\cdot, \sigma)$ for each $\sigma \in K$

From: Hickok, Persistence diagram bundles: A multidimensional generalization of vineyards, 2023.

Persistence Bundle

Definition (Hickok 2023)

- Given $f : K \times B \to \mathbb{R}$
- $E = \{(p, z) \mid p \in B, z \in Dgm_q(f_p)\}$
- $\pi : E \to B$, $\pi(p, z) = p$
- (E, B, π) is the **persistence bundle** of f in degree q

From: Hickok, Persistence diagram bundles: A multidimensional generalization of vineyards, 2023.

Section

Definition

A **local section** of a persistence bundle
(E, B, π) over an open set $U \subseteq B$ is a
continuous map $s : U \to E$ such that
$\pi \circ s = id_U$.
A **(global) section** of a persistence
bundle (E, B, π) is a continuous map
$s : B \to E$ such that $\pi \circ s = id_B$.

From: Hickok, Persistence diagram bundles: A multidimensional generalization of vineyards, 2023.

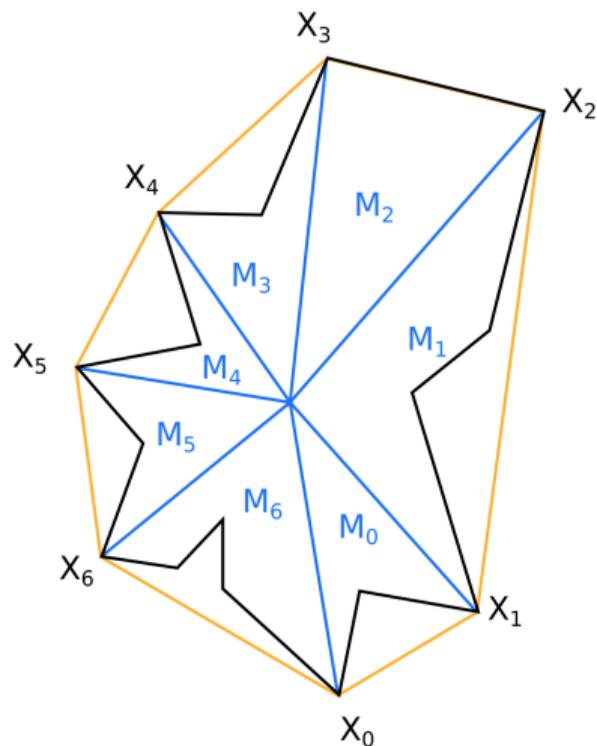# Monodromy definition

Definition

A bundle exhibits non-trivial monodromy if there are no global sections.
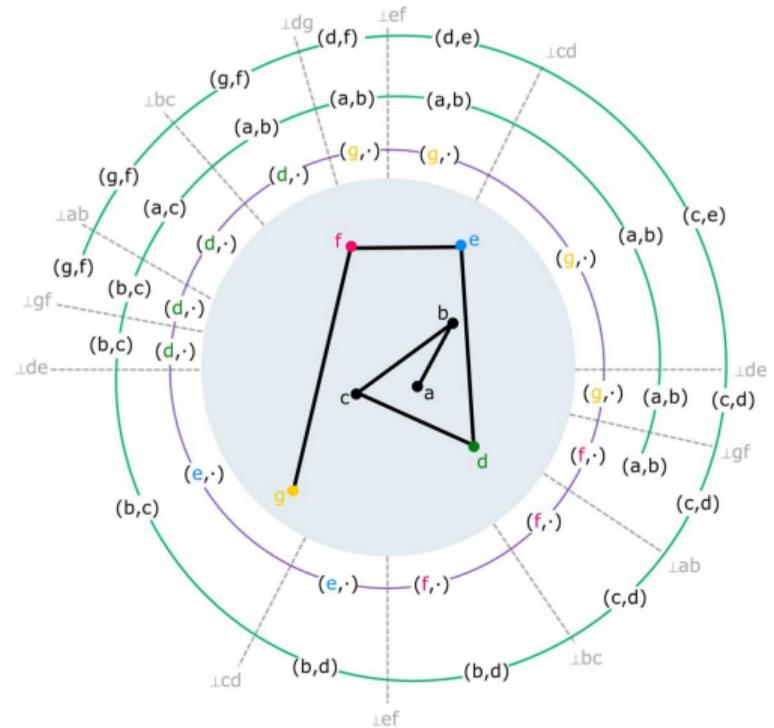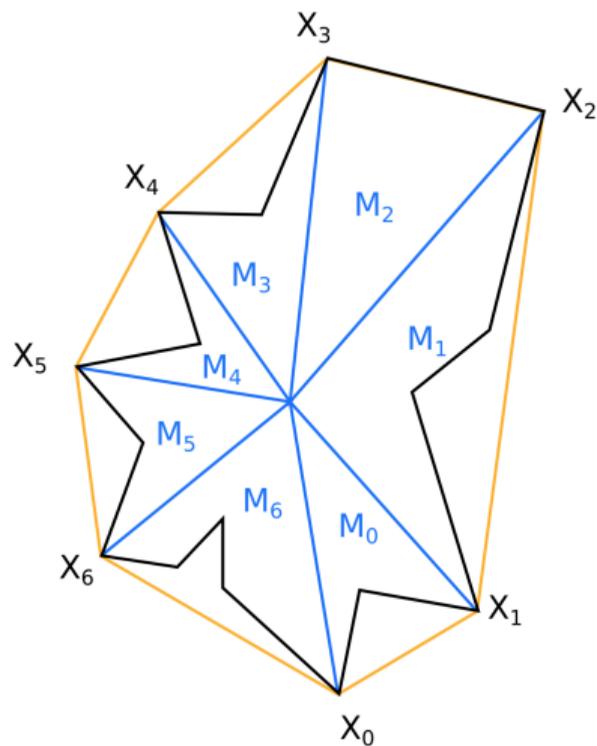
Star-Shaped Spaces

Definition

A set $M \subseteq \mathbb{R}^d$ is **star-shaped** if there exists a point $c \in M$ such that for all $x \in M$, the line segment from c to x is contained in M.

From: Arya et al., Decomposing the Persistent Homology Transform of Star Shaped Objects 2024.

- Assume the convex hull of M, $C(M)$ is a convex polytope (AKA convex hull of finite set of points).
- Boundary of $C(M)$: $\{\sigma_i\}_{i=1}^n$.
- $S_i = C(\sigma_i \cup \{c\})$
- Sector $M_i = S_i \cap M$.

From: Arya et al., Decomposing the Persistent Homology Transform of Star Shaped Objects 2024.
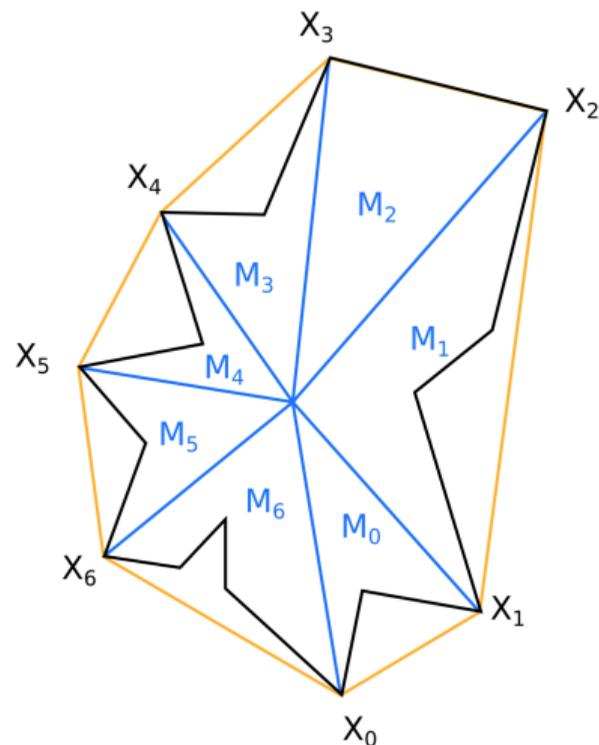
Star-shaped objects have trivial monodromy

Theorem (Arya et al., 2024)

Let $M \subset \mathbb{R}^2$ be a star-shaped object with sectors $\{M^i\}_{i=0}^m$. The for all $w \in \mathbb{S}^1$,

$$\bigoplus_{i=0}^m \widetilde{Pers_0}(M^i, w) \cong \widetilde{Pers_0}(M, w)$$

Theorem (Arya et al., 2024)

Let $M \subset \mathbb{R}^2$ be a star-shaped object. Then the persistence bundle of M in degree 0 has trivial monodromy.
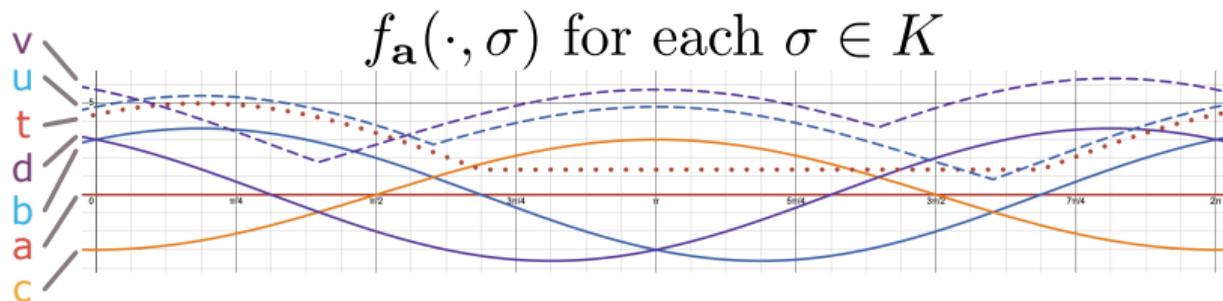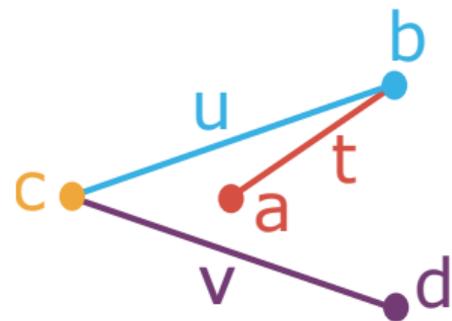
Section 5
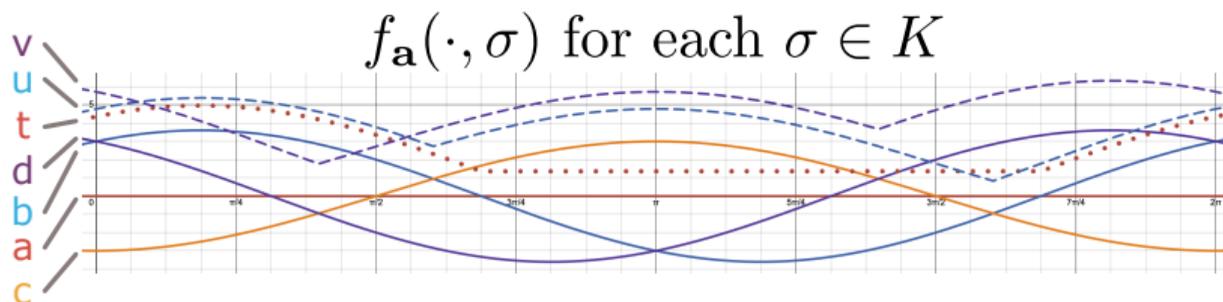
General persistence bundles

Abby's Perturbing a

Definition

- Given $f \times B \to \mathbb{R}$.
- Fix an ordering of the simplices of K, $\{\sigma_1, \ldots, \sigma_n\}$, such that if $\sigma_i < \sigma_j$, then $i < j$.
- Let $A := \{\mathbf{a} \in \mathbb{R}^N \mid a_i \leq a_j, i \leq j\}$
- Define $f_a(\sigma_i, p) = f(\sigma_i, p) + a_i$.

$$f_{\mathbf{a}}(\cdot, \sigma) \text{ for each } \sigma \in K$$

Theorem

Theorem (Hickok 2023)

For a generic fibered filtration function $f : K \times B \to \mathbb{R}$ on a smooth compact manifold B, there is a stratification of the B with finitely many strata such that the entire bundle is determined by the persistent homology at one point per stratum.

$$f_{\mathbf{a}}(\cdot, \sigma) \text{ for each } \sigma \in K$$

Next time

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