Introduction & Topology Basics

Lecture 1 - CMSE 890

Prof. Elizabeth Munch

Michigan State University
::
Dept of Computational Mathematics, Science & Engineering

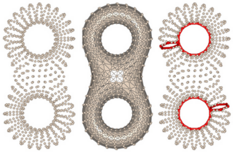
Tues, Aug 26, 2025

Syllabus

- Available on the course website: elizabethmunch.com/cmse890
- Schedule & Office hours
- Slack
- Prerequisites:
  - Linear Algebra
  - Some programming experience
- Python, jupyterhub, and engineering DECS accounts
- Homework
  - Present a problem assigned in the previous class
  - Approximately twice during the semester
  - Goal: Present on something not in your expertise

Textbook

https://www.cs.purdue.edu/homes/tamaldey/book/CTDAbook/CTDAbook.html



Book : Computational Topology for Data Analysis

(To be published by Cambridge University Press)

Tamal K Dey
Department of Computer Science, Purdue University, IN

Yusu Wang
Halicioglu Data Science Institute, UC San Diego, CA

Get an electronic copy

Computational topology has played a synergistic role in bringing together research work from computational geometry, algebraic topology, data analysis, and many other related scientific areas. In recent years, the field has undergone an explosieve growth in the area of data analysis. The application of topological techniques to traditional data analysis, which has earlier developed mostly on a statistical setting, has opened up new opportunities. This book is intended to cover this aspect of computational topology along with the developments of generic techniques for various topology-centered problems. Since the development of persistent homology, the area has grown both in its methodology and applicability. One consequence of this growth has been the development of various algorithms which intertwine with the discoveries of various mathematical structures in the context of processing data. The purpose of this book is to capture these algorithmic developments with the associated mathematical guarantees.

- **Contents**

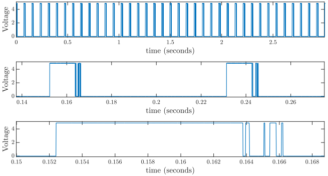
Introductions

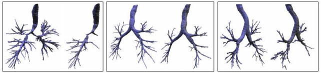
- Name
- Department/Program
- Research interest
- Non-work interest

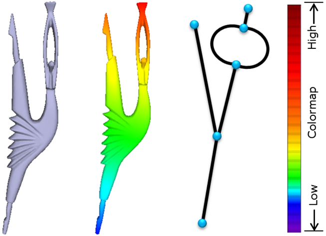
Section 1

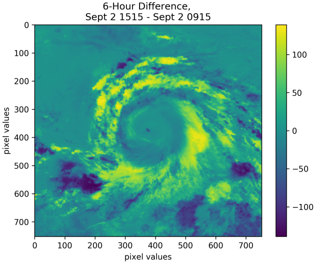
Topological Data Analysis

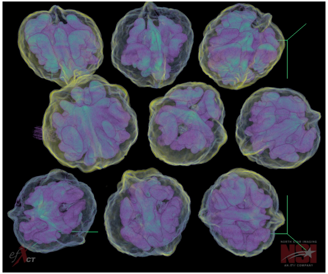
Shape in data







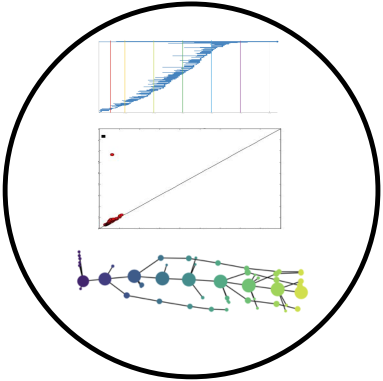




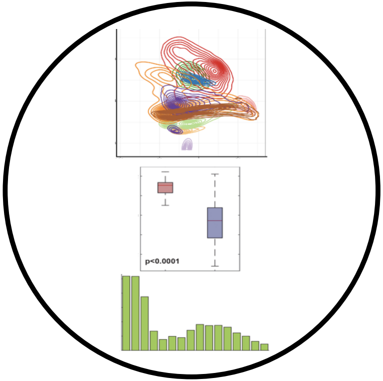
Images: Wikipedia, Szymczak et al., Ma et al.

Topological Data Analysis (TDA)

Raw Data
X-ray CT
Point Clouds
Networks

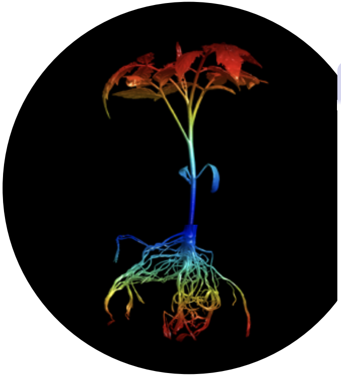


Topological Summary
Persistence Diagrams
Euler Characteristic Curves
Mapper graphs



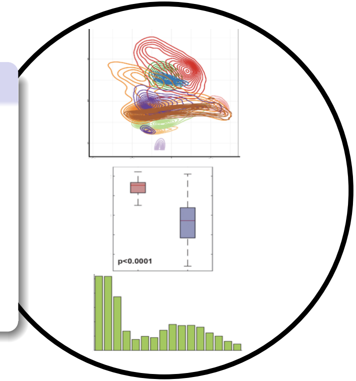
Analysis
Statistics
Machine Learning

Topological Data Analysis (TDA)



Main goal of TDA

Provide
quantifiable,
comparable,
robust,
consise
summaries of the shape of data.



Raw Data
X-ray CT
Point Clouds
Networks

Topological Summary
Persistence Diagrams
Euler Characteristic Curves
Mapper graphs

Analysis
Statistics
Machine Learning

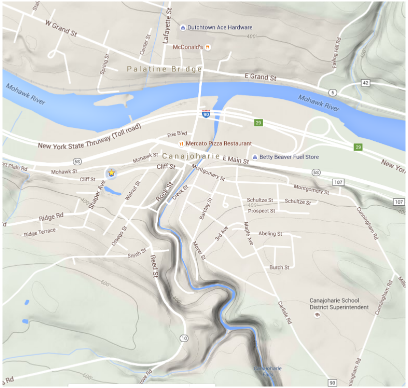
What is topology?

Topology $\neq$ Topography

Mathematical study of spaces preserved under continuous deformations

- stretching and bending
- not tearing or gluing

Study of the shape and features of the surface of the Earth



Images: Wikipedia

History Pt 2

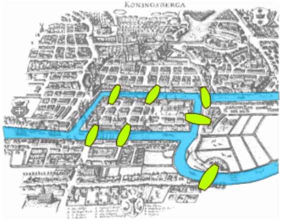
- Esoteric field of study 1700-2000
  - Algebraic topology
  - Applications/intersections with dynamical systems
  - Would never be considered “applied” in traditional sense.

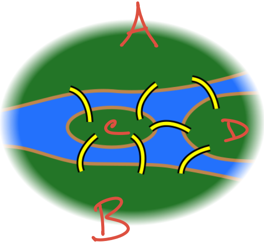
  Topology, the pinnacle of human thought.
  In four centuries it may be useful.
  - Alexander Solzhenitzin, “The First Circle” 1968

- Things change ca.2000
  - Introduction of Persistent Homology

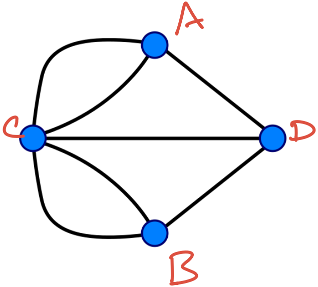
History

Leonhard Euler
(1707-1783)



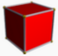
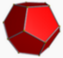


Bridges of Konigsberg



Images: Wikipedia

Topological Invariants- Euler Characteristic

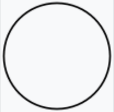
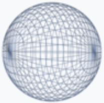
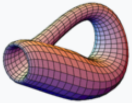
Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $V - E + F$
Tetrahedron		4	6	4	**2**
Hexahedron or cube		8	12	6	**2**
Octahedron		6	12	8	**2**
Dodecahedron		20	30	12	**2**
Icosahedron		12	30	20	**2**

Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $V - E + F$
Tetrahemihexahedron		6	12	7	**1**
Octahemioctahedron		12	24	12	**0**
Cubohemioctahedron		12	24	10	**−2**
Great icosahedron		12	30	20	**2**

$$\text{Vertices} - \text{Edges} + \text{Faces} = \text{Euler Characteristic}$$

Images: Wikipedia

Euler characteristic as topological signature

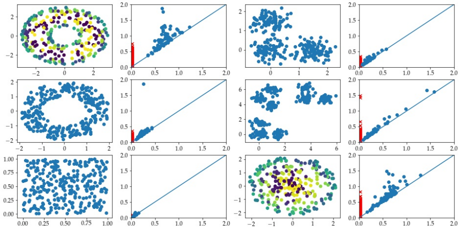
Name	Image	Euler characteristic
Interval		1
Circle		0
Disk		1
Sphere		2
Torus (Product of two circles)		0
Double torus		−2
Triple torus		−4
Real projective plane		1
Möbius strip		0
Klein bottle		0

Different Euler characteristics mean spaces must be topologically **different**

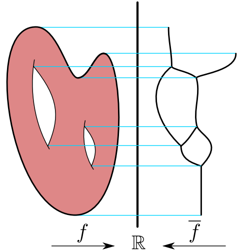
Different spaces might have the **same** Euler characteristics

Euler characteristic is an example of a **topological signature**

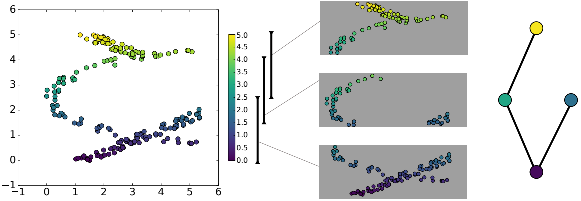
Quantification vs Representation of Shape



Persistent Homology



Reeb graph



Mapper

Current active research directions

- Multidimensional persistence
- Machine learning, statistics
- Time series analysis and dynamical systems
- Metrics
- Parallelization
- Visualization

- Application areas:
  - Neuroscience
  - Plant Biology
  - Gene expression
  - Image processing
  - Sensor networks
  - Atmospheric science

Goals for this course

- Understand the computation and interpretation of several commonly used tools in TDA
  - Persistent Homology
  - Reeb graphs
  - Mapper
- Know what types of data and/or are amenable to TDA methods.
- Have experience working with open-source code banks for computation.

Section 2

Intro to Topology Vocabulary

Goals of this section

- Cover some basic terms from Ch 1.1, 1.2, 1.3
- Depending on your math background, this might be obvious or this might seem impossible. If the latter, spend some time tonight trying some examples! Oh yeah, and read the textbook!

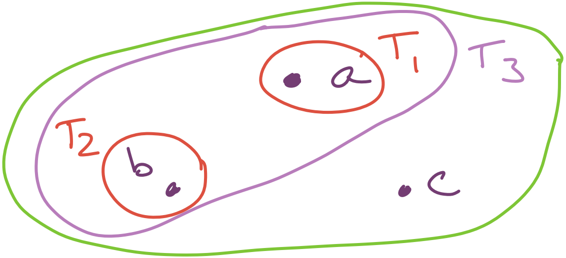
Topology

Definition

A topological space is a point set $\mathbb{T}$ with a set of subsets T (called open sets) such that

- $\underbrace{\emptyset}_{\text{empty set}}, \mathbb{T} \in T$
- For every $U \subseteq T$, the union of the sets in U is in T
- For every finite $U \subseteq T$, the common intersection of the subsets in U is in T.

Ex. $\mathbb{T} = \{a, b, c\}$, $T = \{\emptyset, T_1 = \{a\}, T_2 = \{b\}, T_3 = \{a, b\}, \mathbb{T} = \{a, b, c\}\}$



Metric - L_2

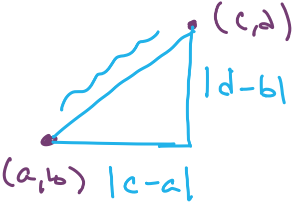
Definition

A metric space is a pair $(\mathbb{T}, d)$ where $\mathbb{T}$ is a set, and $d : \mathbb{T} \times \mathbb{T} \to \mathbb{R}_{\geq 0}$ satisfies

- $d(p, q) = 0$ iff $p = q$
- $d(p, q) = d(q, p)$ for all $p, q \in \mathbb{T}$
- $d(p, q) \leq d(p, r) + d(r, q)$ for all $p, q, r \in \mathbb{T}$

Example: $\mathbb{T} = \mathbb{R}^2$,

$$d((a, b), (c, d)) = \sqrt{(c - a)^2 + (b - d)^2}$$



Metric - L_∞

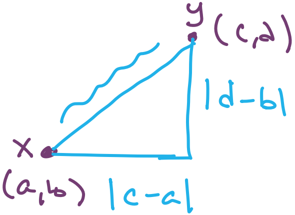
Definition

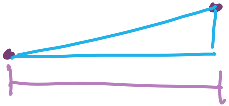
A metric space is a pair $(\mathbb{T}, d)$ where $\mathbb{T}$ is a set, and $d : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ satisfies

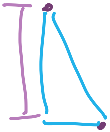
- $d(p, q) = 0$ iff $p = q$
- $d(p, q) = d(q, p)$ for all $p, q \in \mathbb{T}$
- $d(p, q) \leq d(p, r) + d(r, q)$ for all $p, q, r \in \mathbb{T}$

Example: $\mathbb{T} = \mathbb{R}^2$,

$$d((a, b), (c, d)) = \max\{|u_1 - v_1|, |u_2 - v_2|\}$$







Metric Topology

Definition

Given a metric space $(\mathbb{T}, d)$, an open metric ball is

$$\underbrace{B_o(c, \overset{\text{radius}}{r})}_{\text{center}} = \{p \in \mathbb{T} \mid d(p, c) < r\}.$$

The metric topology is the set of all metric balls

$$\{B_o(c, r) \mid c \in \mathbb{T}, 0 < r \leq \infty\}.$$

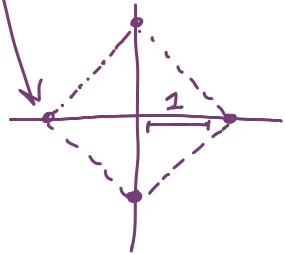
Ex. $\mathbb{R}$, $\mathbb{R}^2$

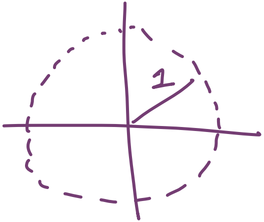
$\hookrightarrow (\mathbb{R}, \|\cdot\|_2)$

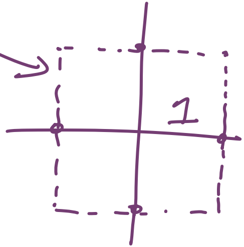
TRY IT:

Draw the subset of $\mathbb{R}^2$ contained in $B_o(0,1)$ for $d((u_1,u_2),(v_1,v_2)) =$

- $\|u-v\|_1 = |u_1 - v_1| + |u_2 - v_2|$
- $\|u-v\|_2 = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$
- $\|u-v\|_\infty = \max\{|u_1 - v_1|, |u_2 - v_2|\}$





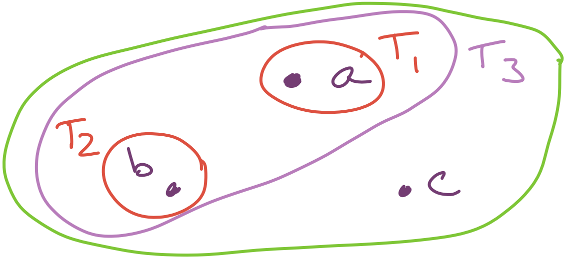


Open and closed sets

Definition

A set is open if it is in the topology T. A set is closed if its complement is open.

Ex 1. $\mathbb{T} = \{a, b, c\}$, $T = \{\emptyset, T_1 = \{a\}, T_2 = \{b\}, T_3 = \{a, b\}, \mathbb{T} = \{a, b, c\}\}$



example

$$\{c\} = T_3^{\,c}$$
$$= \mathbb{T} \setminus T_3$$
$$= \{a,b,c\} \setminus \{a,b\}$$

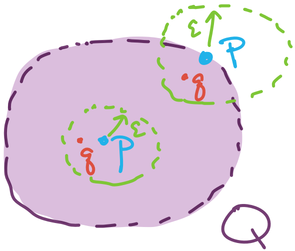
Open and closed sets - metric space version

Limit points

$\mathbb{R}^2$

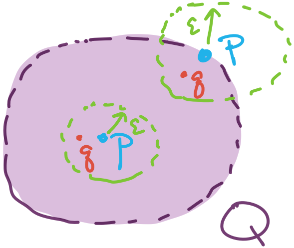
Definition

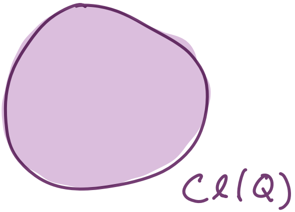
Let $Q \subset \mathbb{T}$ be a point set. A point $p \in \mathbb{T}$ is a *limit point* of Q if for every $\varepsilon > 0$, Q contains a point $q \neq p$ with $d(p, q) < \varepsilon$. $q \in Q$



Definition

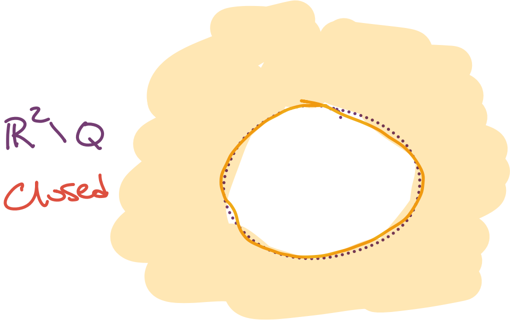
- $\mathrm{Cl}(Q)$: The closure of a point set $Q \subseteq \mathbb{T}$ is the set containing every point in Q and every limit point of Q.
- A point set Q is closed if $Q = \mathrm{Cl}(Q)$, i.e. Q contains all its limit points.





Definition

- The complement of a point set Q is $T \setminus Q$.
- A point set Q is open if its complement is closed, i.e. $T \setminus Q = \mathrm{Cl}(T \setminus Q)$.

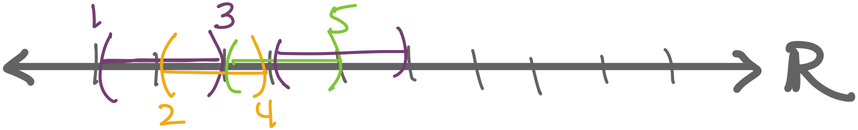


Open cover

Definition

An open (closed) cover of a topological space $(\mathbb{T}, T)$ is a collection C of open (closed) sets so that $T \subseteq \bigcup_{U \in C}$.

Ex. $\mathbb{R}$, $C = \{(n-1, n+1) \mid n \in \mathbb{Z}\}$



Connected

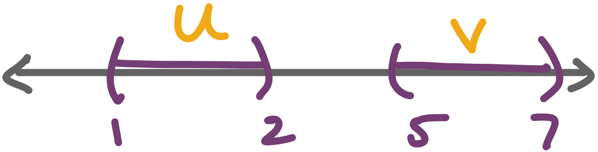
no point in both

Definition

A topological space $(\mathbb{T}, T)$ is disconnected if there are two disjoint non-empty open sets $U, V \in T$ so that $\mathbb{T} = U \cup V$. A topological space is connected if its not disconnected.

Ex. $A = (1,2) \cup (5,7) \subset \mathbb{R}$

$A = U \cup V$



Section 3

Maps

Definition

$f(t) \in U$

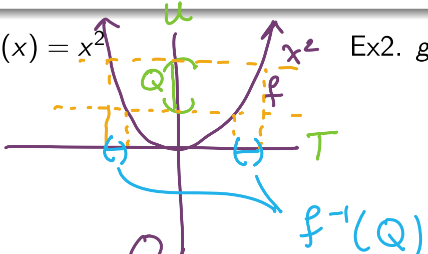
A function $f : T \to U$ is continuous if for every open set $Q \subseteq U$, $\underbrace{f^{-1}(Q)}$ is open. Continuous functions are also called maps.

$f^{-1}(Q)$

$= \{ t \in T \mid f(t) \in Q \}$

Ex1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

$x \mapsto x^2$



Ex2. $g : \mathbb{R} \to \mathbb{R}$ $g(x) = \lfloor x \rfloor$

$f : T \to U \supseteq Q$

$t \mapsto f(t)$

Embedding

Definition

A map $g : T \to U$ is an embedding of T into U if g is injective.
Injective (1-1): $f(x) = f(y)$ *iff* $x = y$

Ex1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$

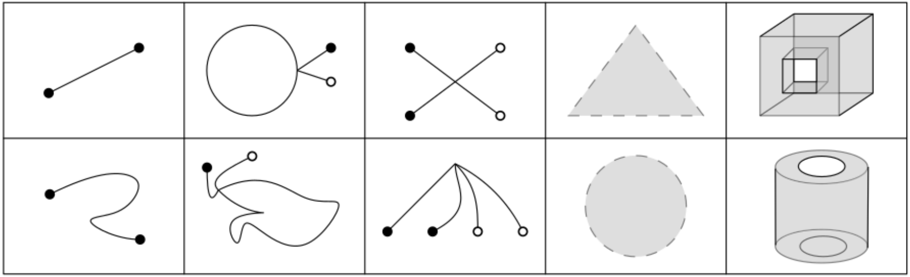
Start here Thursday!

Ex2. $g : \mathbb{S}^1 \to \mathbb{R}^2$

Homeomorphism

Definition

Let T and U be topological spaces.
A *homeomorphism* is a bijective map $h : T \to U$ whose inverse is also continuous.
Two topological spaces are *homeomorphic* if there exists a homeomorphism between them.



Homeomorphism: Cheap trick

Proposition

If T and U are compact metric spaces, every bijective map from T to U has a continuous inverse.

Homework for next time

Need a volunteer! For this homework, it can't be someone who is a Math PhD student, preferably someone who hasn't taken a topology class.

Choose two of the following to present.

1. DW 1.6.1. Be sure to explain *why* the constructions you have created are/are not Hausdorff. (Sona)
2. DW 1.6.2
3. DW 1.6.3
4. DW 1.6.4
5. DW 1.6.5
6. DW 1.6.6