Here Comes the Homology

Lecture 5 - CMSE 890

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Goals

Goals for today:

- Ch 2.4: Cycles and Boundaries

Section 1

Linear Algebra Review & Homology

Definition

A field $(k, +, \cdot)$ is a set k with two operations $+$ and $\cdot$ such that for any $a, b, c \in k$:

- Closure:
- Commutativity:
- Associativity:
- Identity:
- Inverses:
- Distributivity:

Examples:

Vector space

Definition

A vector space over a field k is a set V with vector addition and scalar multiplication such that

- Associative $+$:
- Commutative $+$:
- Identity
  - $+$
  - $\cdot$
- Inverses:
  - $+$

- Scalar vs. field mult:
- Distributivity:
  -
  -

Examples:

Basis

Definition

A basis for a vector space V is a collection of vectors $\{b_\alpha\}_{\alpha \in A}$ such that

- they are linearly independent and,
- they span V.

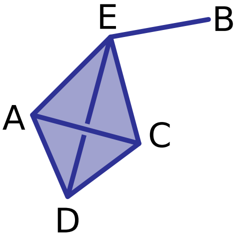
Goal

Build a vector space from a simplicial complex!

p-Chains

Let K be a simplicial complex and fix a dimension p.
A p-chain is a finite formal sum of p-simplices in K, written

$$\alpha = \sum a_i \sigma_i$$



Addition of chains

p-chains are added component-wise: if $\alpha = \sum a_i\sigma_i$ and $\beta = \sum b_i\sigma_i$, then

$\alpha + \beta =$

Chain group

The collection of p-chains with addition is called the p^{th}-chain group (vector space), $C_p(K)$.

Some checks

- Associative $+$:
- Commutative $+$:
- Zero element
- Inverses:

Linear Transformations

A linear transformation between two vector spaces V and W is a map $T : V \to W$ such that the following hold:

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Matrix representation:

Boundary maps[1]

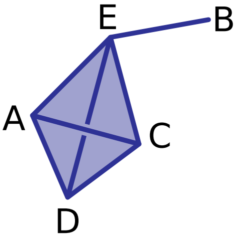
$$\begin{array}{rccc} \partial_p : & C_p(K) & \to & C_{p-1}(K) \\ & \sigma = [v_0, \cdots, v_p] & \mapsto & \sum_{j=0}^{p} [v_0, \cdots, \widehat{v_j}, \cdots v_p] \end{array}$$

[1] Warning: We are assuming $\mathbb{Z}_2$ coefficients from now on!

Checking linearity

$$\partial_p(\alpha+\beta) = \partial_p(\alpha) + \partial_p(\beta); \qquad \alpha = \sum a_i \sigma_i, \qquad \beta = \sum b_i, \sigma_i$$

Homework



- $\partial_1([a, e]) =$
- $\partial_1([a, e] + [b, e]) =$
- $\partial_1([a, e] + [c, e] + [c, d] + [a, d]) =$
- $\partial_2([a, c, e] + [a, c, d]) =$