

Directional Transform and the Euler Characteristic Transform

Lecture 20 - CMSE 890

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- Euler Characteristic Curves and the Euler Characteristic
- Reading: *An Invitation to the Euler Characteristic Transform*,
<https://doi.org/10.1080/00029890.2024.2409616>

Section 1

Euler Characteristic Curve

Topological Invariants- Euler Characteristic

Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $V - E + F$
Tetrahedron		4	6	4	2
Hexahedron or cube		8	12	6	2
Octahedron		6	12	8	2
Dodecahedron		20	30	12	2
Icosahedron		12	30	20	2

Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $V - E + F$
Tetrahemihexahedron		6	12	7	1
Octahemioctahedron		12	24	12	0
Cubohemioctahedron		12	24	10	-2
Great icosahedron		12	30	20	2

$$\text{Vertices} - \text{Edges} + \text{Faces} = \text{Euler Characteristic}$$

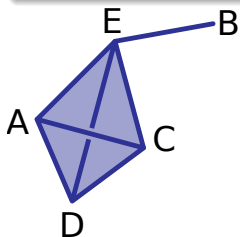
Images: Wikipedia

Generalized Euler Characteristic

Definition

For any simplicial complex K where n_p is the number of p -dimensional simplices, the Euler characteristic is

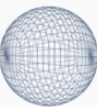
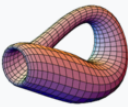
$$\chi(K) = \sum_p (-1)^p n_p$$



$$\begin{aligned} n_0 &= 5 \\ n_1 &= 7 \\ n_2 &= 4 \\ n_3 &= 0 \end{aligned}$$

$$5 - 7 + 4 - 0 = 2$$

Euler characteristic of spaces

Name	Image	Euler characteristic
Interval		1
Circle		0
Disk		1
Sphere		2
Torus (Product of two circles)		0
Double torus		-2
Triple torus		-4
Real projective plane		1
Möbius strip		0
Klein bottle		0

Different Euler characteristics mean spaces must be topologically **different**

Different spaces might have the **same** Euler characteristics

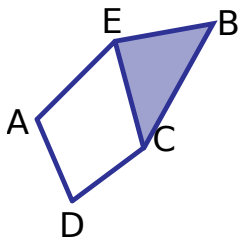
Euler characteristic is an example of a **topological signature**

Euler characteristic and Betti numbers

Theorem

For any simplicial complex where n_p is the number of p -dimensional simplices,

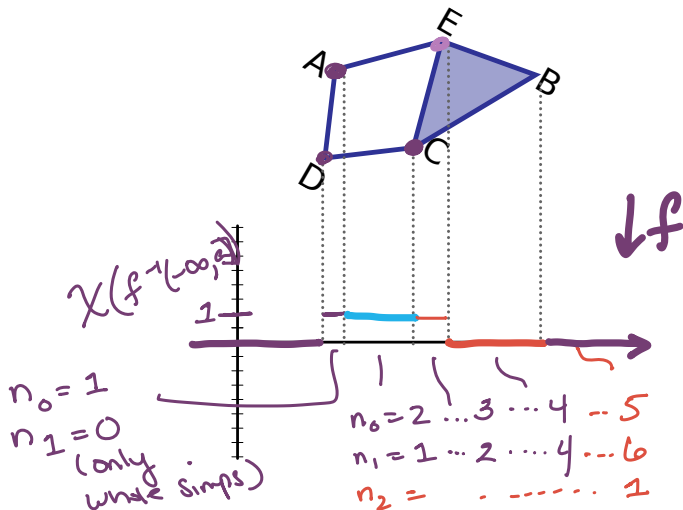
$$\chi = \sum_p (-1)^p n_p = \sum_p (-1)^p \beta_p \leftarrow \text{Betti \#} \\ \text{dim } (H_p(K))$$



$$\begin{array}{ccccccc} n_2 & - & n_1 & + & n_0 & = & \beta_2 & - & \beta_1 & + & \beta_0 \\ 1 & - & 6 & + & 5 & & 0 & - & 1 & + & 1 \\ & & & & & & \parallel & & \parallel & & \\ & & & & & & 0 & & 0 & & \end{array}$$

Euler Characteristic Curve

$$E(a) = \chi(f^{-1}(-\infty, a])$$

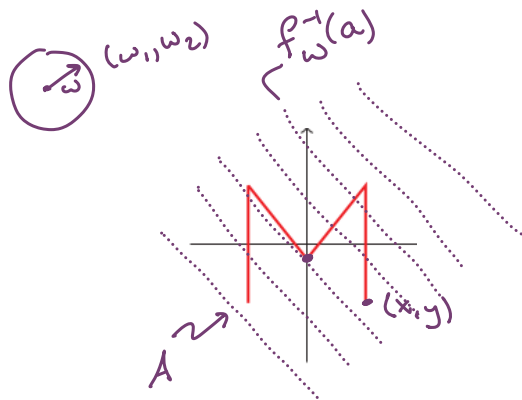


Section 2

Directional Transform

Direction induces a function

- Fix a subset of $\mathbb{R}^d$, A
- Fix a direction $\omega \in \mathbb{S}^{d-1}$
- $f_\omega : A \rightarrow \mathbb{R}$, $f_\omega(x) = \langle x, \omega \rangle$



Function induces an algebraic representation

- χ_ω = Euler characteristic curve of f_ω
- $Pers_{k,\omega}$ = k -dimensional persistence diagram of f_ω

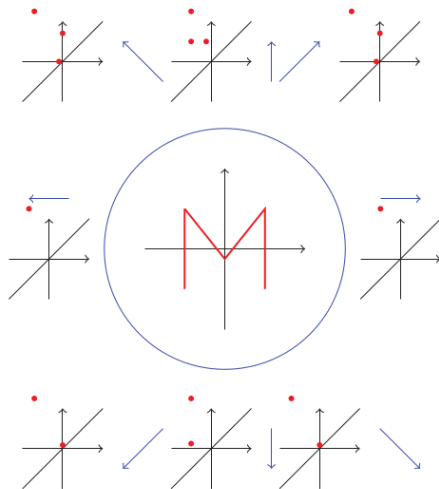


Figure from: Turner, Mukherjee, Boyer. Persistent homology transform for modeling shapes and surfaces.

The full transform(s)

$$\begin{array}{ccc} PHT(A) : \mathbb{S}^{d-1} & \longrightarrow & \mathcal{Dgm}^d \\ \omega & \longmapsto & (Pers_{0,\omega}(A), \dots, Pers_{d-1,\omega}(A)) \end{array}$$

$$\begin{array}{ccc} ECT(A) : \mathbb{S}^{d-1} & \longrightarrow & \text{Functions on } \mathbb{R} \\ \omega & \longmapsto & \chi_\omega(A) \\ & & a \longmapsto \chi(f_\omega^{-1}(-\infty, a]) \end{array}$$

Meta transforms

$\mathcal{M}_d$ = space of all finite simplicial complexes embedded in $\mathbb{R}^d$

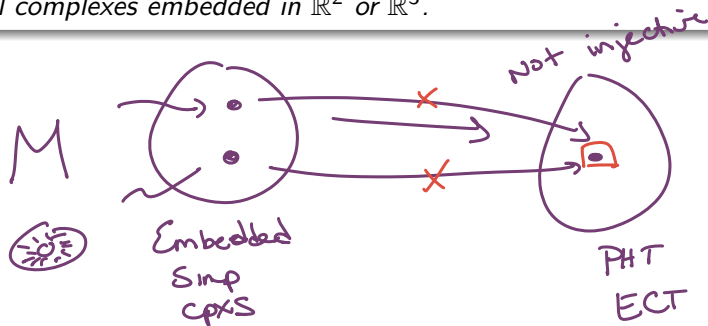
$$\begin{aligned} PHT : \mathcal{M}_d &\longrightarrow \text{Functions from } \mathbb{S}^{d-1} \text{ to } \mathcal{Dgm}^d \\ A &\longmapsto \omega \mapsto (Pers_{0,\omega}(A), \dots, Pers_{d-1,\omega}(A)) \end{aligned}$$

$$\begin{aligned} ECT : \mathcal{M}_d &\longrightarrow \text{Functions from } \mathbb{S}^{d-1} \text{ to (Functions on } \mathbb{R}) \\ A &\longmapsto \omega \mapsto \chi_\omega(A) \end{aligned}$$

Turner's theorem

Theorem (Turner, Mukherjee, Boyer)

The ECT and PHT are injective on the space $\mathcal{M}_d$ of finite simplicial complexes embedded in $\mathbb{R}^2$ or $\mathbb{R}^3$.



Idea: The ECT, PHT are faithful rep's of input shape.

Generalizations by Ghrist et al; Curry et al

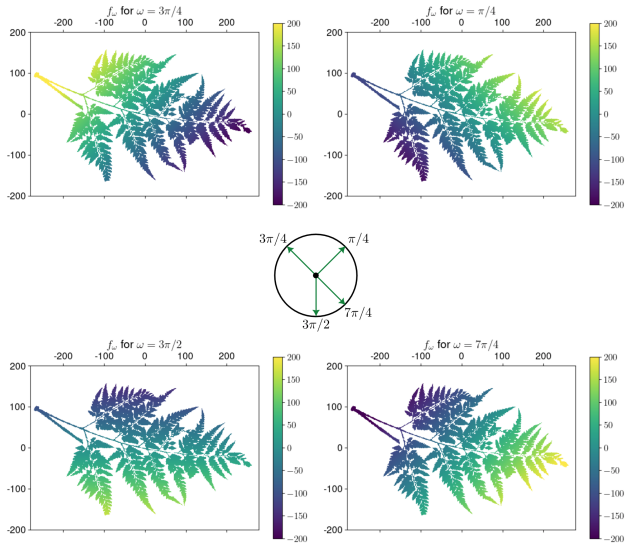
Section 3

The ECT

Fern example



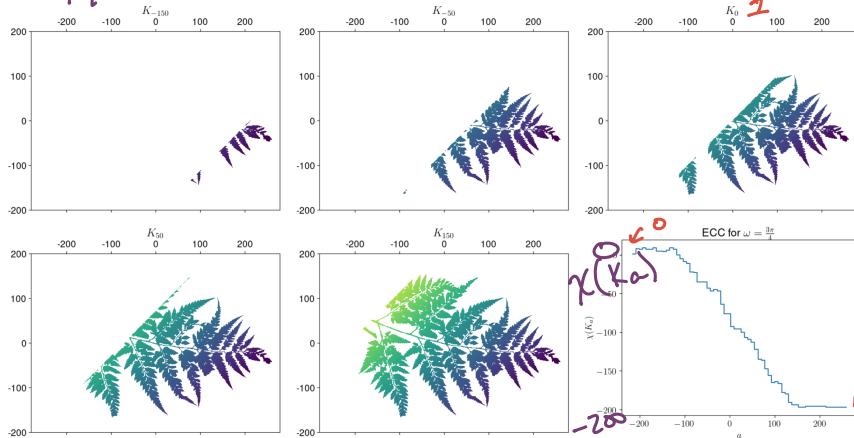
Fern - Directional Transform



$$f_\omega(x) = \langle x, \omega \rangle$$

Fern - Thresholds

$$\omega = 3\pi/4$$



$\beta_0 - \beta_1 = \text{big}$

$\chi(K_\omega)$

$\chi(K)$

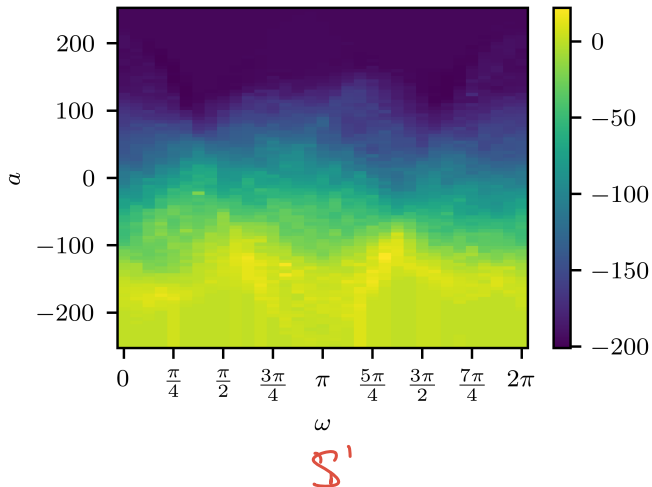
$$ECC(a) = \chi(f_\omega^{-1}(-\infty, a])$$

$$K_a = f_\omega^{-1}(-\infty, a]$$



$$ECT(\omega, a) = \chi(f_{\omega}^{-1}(-\infty, a])$$

ECT of Fern Leaf



Next time

