Nerve, Cech, and Rips Complexes

Lecture 4 - CMSE 890

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Section 2.2 Goals

Goals for today:

- Complexes from point clouds: Rips & Čech complexes
- Sparse Complexes: Alpha complex

Recall: Geometric vs Abstract Simplicial Complex

- Given a collection of points $V \subseteq \mathbb{R}^N$
- For a subset of these $\{a_0, \ldots, a_n\}$, a (geometric) n-simplex is the convex hull of the points.
- A simplicial complex is a collection of geometric n simplices so that
  - Every face of a simplex is also in the complex.
  - The intersection of any two simplices is either empty or a face of both.

- Given a finite set V
- An abstract simplex is a subset of V.
- An abstract simplicial complex is a set K of finite subsets of some V such that if $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$.

Section 1

Čech and Rips Complexes

Point cloud

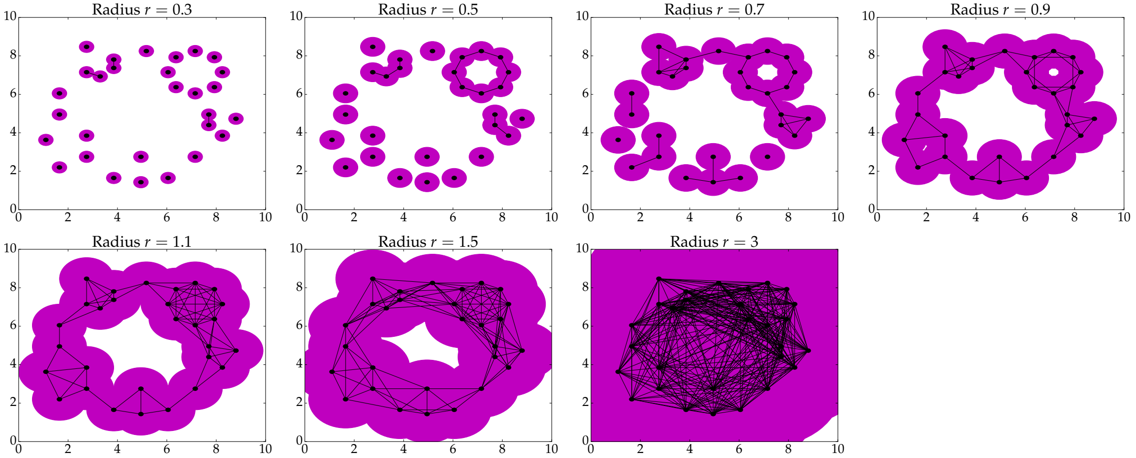
A point cloud is a (finite) collection of points in a metric space (M, d).

Ball

$$B_o(x, r) = \{y \in M \mid d(x, y) < r\}$$

$$B(x, r) = \{y \in M \mid d(x, y) \leq r\}$$

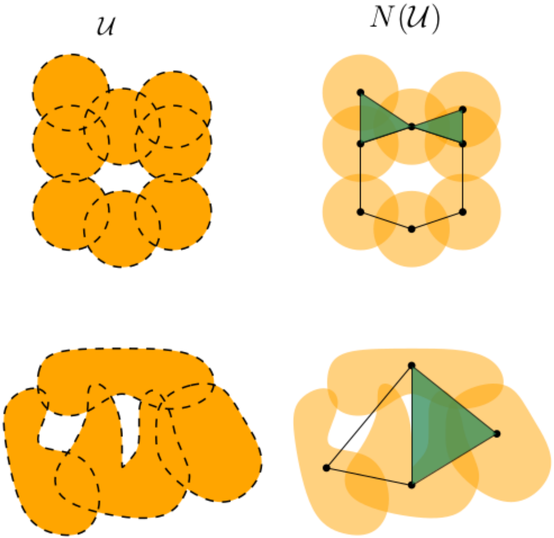
What we want to study



From last time: Nerve

Given a finite collection of sets $\mathcal{F}$, the **nerve** is

$$\mathrm{Nrv}(\mathcal{U}) = \{X \subseteq \mathcal{F} \mid \bigcap_{U \in X} U \neq \emptyset\}.$$



From earlier: Homotopy type

Definition

Two topological spaces T and U are homotopy equivalent if there exist maps $g : T \to U$ and $h : U \to T$ such that $h \circ g$ and $g \circ h$ are homotopic to the appropriate identity maps.

- *Intuition*: Can deform one space into the other.
- Example: Divide the alphabet into *equivalence classes*: collections of letters that are all homotopy equivalent to every other letter in their collection.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

Semi-covered in lecture 2

Nerve lemma (Metric space version)

Theorem

Given a finite cover $\mathcal{U}$ (open or closed) of a metric space M, the underlying space $|N(\mathcal{U})|$ is homotopy equivalent to M if every non-empty intersection $\bigcap_{i=0}^{k} U_{\alpha_i}$ of cover elements is homotopy equivalent to a point (contractible).

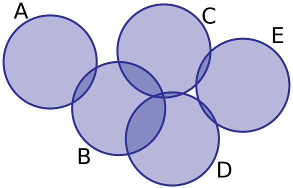
Čech complex

Definition

Let $P \subset (M, d)$ be a finite point cloud. Fix $r \geq 0$. The Čech complex is

$$\check{C}^r(P) = \left\{ \sigma \subseteq P \mid \bigcap_{x \in \sigma} B(x,r) \neq \emptyset \right\}$$
$$= \mathrm{N}(\{B(x,r)\}_{x \in P})$$

Example: Cech complex



Warning

The Čech complex is an abstract simplicial complex. The obvious map into $\mathbb{R}^N$ doesn't necessarily get you a geometric simplicial complex!

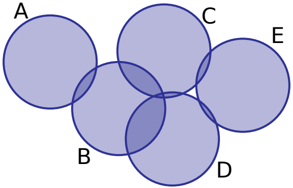
Rips complex

Definition

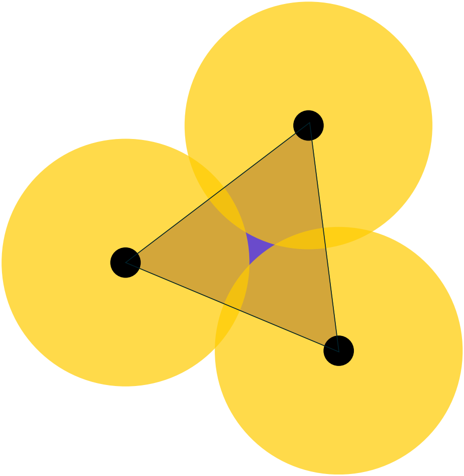
Given $P \subseteq (M, d)$, the Vietoris-Rips complex is

$$VR^r(P) = \{\sigma \subseteq P \mid d(x_i, x_j) \leq 2r \text{ for all } x_i, x_j \in \sigma\}$$

Example: Rips complex



Equilateral triangle example



Rips-Čech Lemma

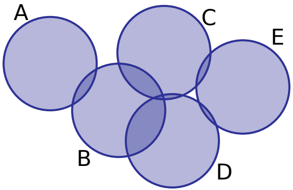
Theorem

Given point cloud $P \subset (M, d)$ *and* $r \geq 0$,

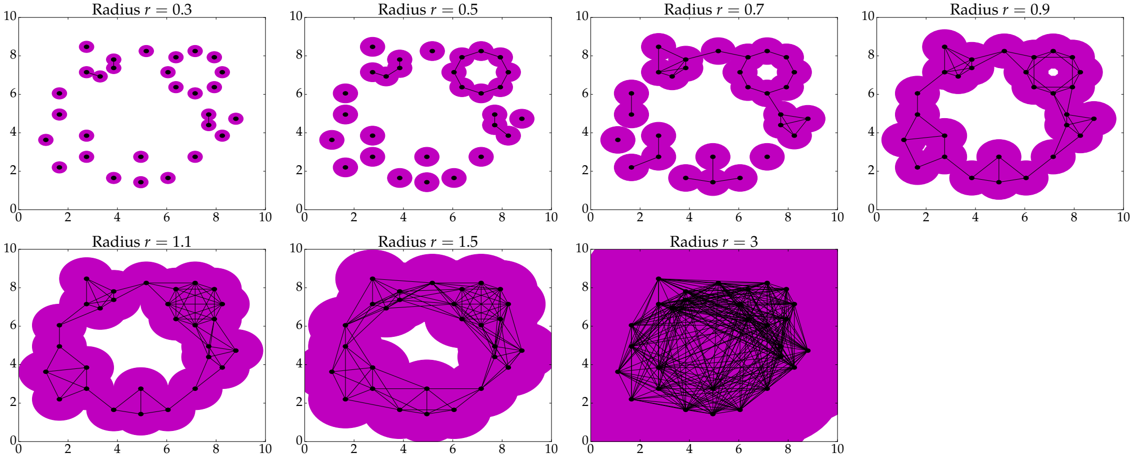
$$\check{C}^r(P) \subseteq VR^r(P) \subseteq \check{C}^{2r}(P)$$

Warning: Radius vs diameter

$$VR^r(P) = \{\sigma \subseteq P \mid d(x_i, x_j) \leq 2r \text{ for all } x_i, x_j \in \sigma\}$$



What we want to study



Section 2

Alpha complex

Voronoi diagram

Given a point cloud $P \subseteq \mathbb{R}^N$.
The Voronoi cell of $u \in P$ is

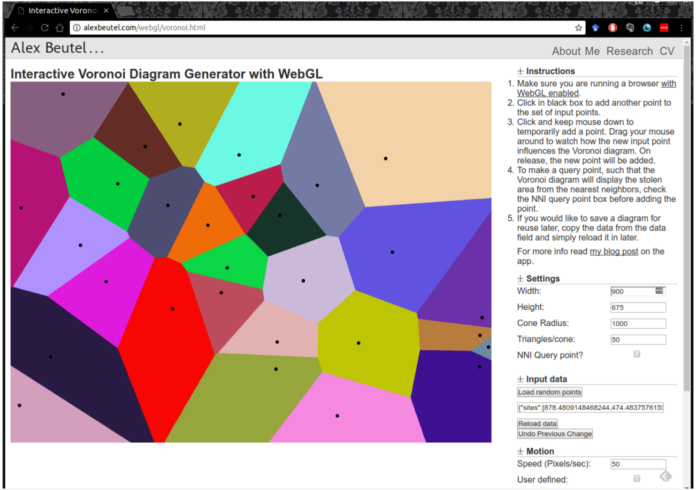
$$V_u = \{x \in \mathbb{R}^d \mid \|x - u\| \leq \|x - v\|, v \in P\}$$

The Voronoi diagram is the collection of Voronoi cells $Vor(P) = \{V_u \mid u \in P\}$.

Simple examples

Website

`http://alexbeutel.com/webgl/voronoi.html`

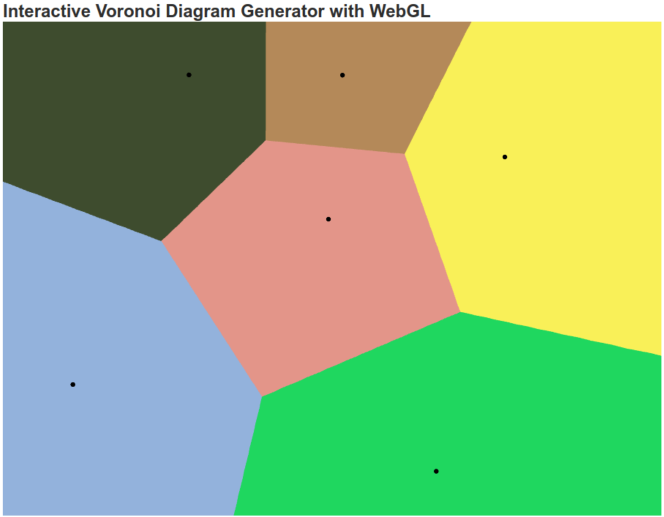


Delaunay triangulation

The Delaunay complex of point cloud $P \subseteq \mathbb{R}^N$ is the nerve of the Voronoi diagram.

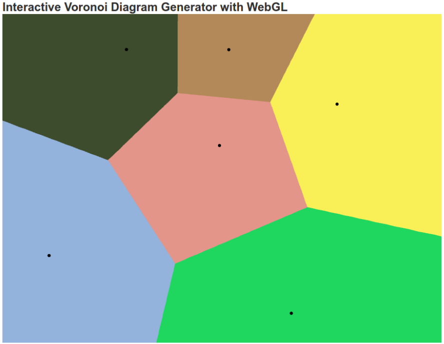
$$\mathrm{Del}(P) = \{\sigma \subseteq P \mid \bigcap_{u \in P} V_u \neq \emptyset\}.$$

Example

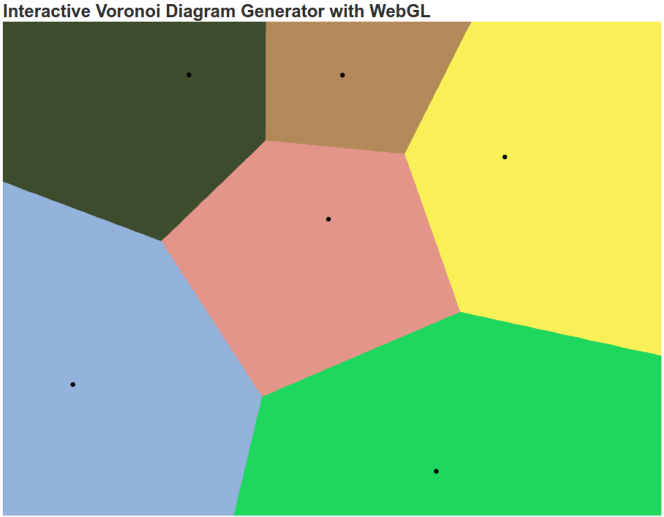


Properties

- Delaunay is an abstract simplicial complex.
- If we have points in general position, the obvious embedding gives a geometric simplicial complex.
- Delaunay is FIXED (has nothing to do with a radius or diameter parameter....)



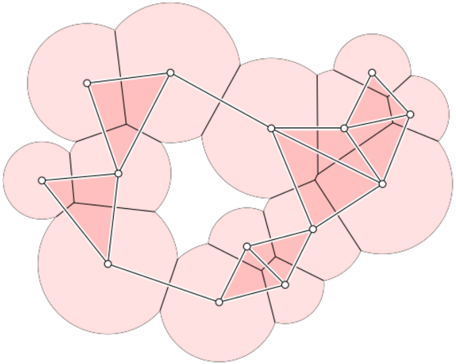
Leading up to the alpha complex



Alpha complex

$\mathrm{Del}_p^\alpha = \{x \in B(p,\alpha) \mid d(x,p) \leq d(x,q) \text{ for all } q \in P\} = B(p,r) \cap V_p$
The alpha complex for point cloud P with radius $r \geq 0$ is the nerve

$$\mathrm{Del}_p^\alpha = \mathrm{Nrv}(\{D_p^\alpha \mid p \in P\}).$$



Properties

- $\mathrm{Alpha}(r) \subseteq \mathrm{Delaunay}$
- $\mathrm{Alpha}(r) \subseteq \check{C}(r)$
- $\mathrm{Alpha}(r)$ has the same homotopy type as the union of balls of radius r.

For next time

- EH III.7 (p75) Let $P \subseteq \mathbb{R}^d$ be a finite set of points in general position. Denote by $\check{C}(r)$ and $\mathrm{Alpha}(r)$ as the Čech and alpha complexes for radius $r \geq 0$, respectively.

  Is it true that $\mathrm{Alpha}(r) = \check{C}(r) \cap \mathrm{Delaunay}$*?*

  If yes, prove the following two subcomplex relations. If no, give examples to show which subcomplex relations are not valid.

  1. $\mathrm{Alpha}(r) \subseteq \check{C}(r) \cap \mathrm{Delaunay}$.
  2. $\check{C}(r) \cap \mathrm{Delaunay} \subseteq \mathrm{Alpha}(r)$