# Directional Transform and the Euler Characteristic Transform

## Lecture 20 - CMSE 890

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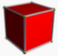
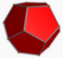
# Plan

- Euler Characteristic Curves and the Euler Characteristic
- Reading: *An Invitation to the Euler Characteristic Transform*, `https://doi.org/10.1080/00029890.2024.2409616`

Section 1

# Euler Characteristic Curve

# Topological Invariants- Euler Characteristic

| Name | Image | Vertices $V$ | Edges $E$ | Faces $F$ | Euler characteristic: $V-E+F$ |
|---|---|---|---|---|---|
| Tetrahedron |  | 4 | 6 | 4 | **2** |
| Hexahedron or cube |  | 8 | 12 | 6 | **2** |
| Octahedron |  | 6 | 12 | 8 | **2** |
| Dodecahedron |  | 20 | 30 | 12 | **2** |
| Icosahedron |  | 12 | 30 | 20 | **2** |

| Name | Image | Vertices $V$ | Edges $E$ | Faces $F$ | Euler characteristic: $V-E+F$ |
|---|---|---|---|---|---|
| Tetrahemihexahedron |  | 6 | 12 | 7 | **1** |
| Octahemioctahedron |  | 12 | 24 | 12 | **0** |
| Cubohemioctahedron |  | 12 | 24 | 10 | **−2** |
| Great icosahedron |  | 12 | 30 | 20 | **2** |

$$\text{Vertices} - \text{Edges} + \text{Faces} = \text{Euler Characteristic}$$

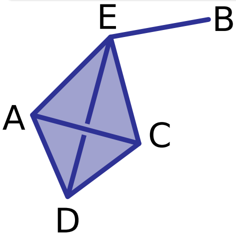
Images: Wikipedia

# Generalized Euler Chacteristic

## Definition

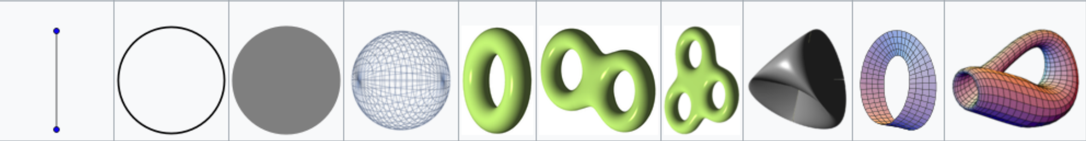
For any simplicial complex $K$ where $n_p$ is the number of $p$-dimensional simplices, the Euler characteristic is

$$\chi(K) = \sum_p (-1)^p n_p$$



# Euler characteristic of spaces

| Name | Interval | Circle | Disk | Sphere | Torus (Product of two circles) | Double torus | Triple torus | Real projective plane | Möbius strip | Klein bottle |
|---|---|---|---|---|---|---|---|---|---|---|
| Euler characteristic | 1 | 0 | 1 | 2 | 0 | −2 | −4 | 1 | 0 | 0 |



**Different** Euler characteristics mean spaces must be topologically **different**

**Different** spaces might have the **same** Euler characteristics

Euler characteristic is an example of a **topological signature**

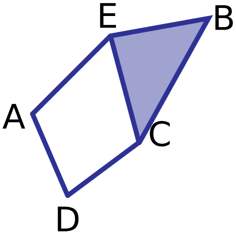
# Euler characteristic and Betti numbers

**Theorem**

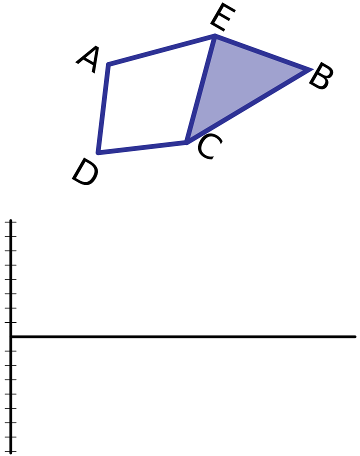
*For any simplicial complex where $n_p$ is the number of $p$-dimensional simplices,*

$$\sum_p (-1)^p n_p = \sum_p (-1)^p \beta_p$$

$$n_2 - n_1 + n_0 = \beta_2 - \beta_1 + \beta_0$$



# Euler Characteristic Curve



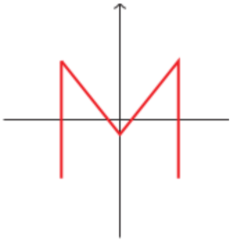

$$E(a) = \chi(f^{-1}(-\infty, a])$$

Section 2

# Directional Transform

# Direction induces a function

- Fix a subset of $\mathbb{R}^d$, $A$
- Fix a direction $\omega \in \mathbb{S}^{d-1}$
- $f_\omega : A \to \mathbb{R}$, $f_\omega(x) = \langle x, \omega \rangle$



# Function induces an algebraic representation

- $\chi_\omega$ = Euler characteristic curve of $f_\omega$
- $Pers_{k,\omega}$ = $k$-dimensional persistence diagram of $f_\omega$

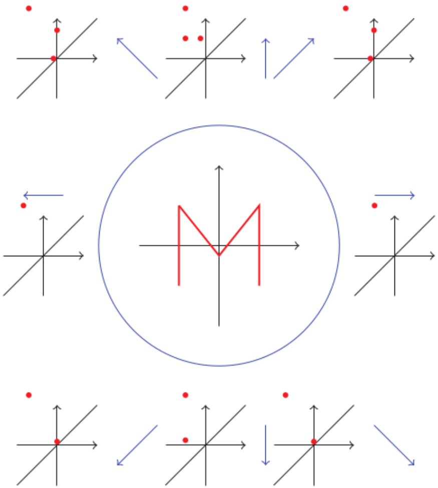


Figure from: Turner, Mukherjee, Boyer. Persistent homology transform for modeling shapes and surfaces.

# The full transform(s)

$$\begin{aligned} PHT(A): \quad \mathbb{S}^{d-1} &\longrightarrow \mathcal{D}gm^d \\ \omega &\longmapsto (Pers_{0,\omega}(A), \ldots, Pers_{d-1,\omega}(A)) \end{aligned}$$

$$\begin{aligned} ECT(A): \quad \mathbb{S}^{d-1} &\longrightarrow \text{Functions on } \mathbb{R} \\ \omega &\longmapsto \chi_\omega(A) \end{aligned}$$

# Meta transforms

$\mathcal{M}_d$ = space of all finite simplicial complexes embedded in $\mathbb{R}^d$

$$\begin{array}{rccc} PHT: & \mathcal{M}_d & \longrightarrow & \text{Functions from } \mathbb{S}^{d-1} \text{ to } \mathcal{D}gm^d \\ & A & \longmapsto & \omega \mapsto (Pers_{0,\omega}(A), \ldots, Pers_{d-1,\omega}(A)) \end{array}$$

$$\begin{array}{rccc} ECT: & \mathcal{M}_d & \longrightarrow & \text{Functions from } \mathbb{S}^{d-1} \text{ to (Functions on } \mathbb{R}) \\ & A & \longmapsto & \omega \mapsto \chi_\omega(A) \end{array}$$

# Turner's theorem

**Theorem (Turner, Mukherjee, Boyer)**

*The ECT and PHT are injective on the space $\mathcal{M}_d$ of finite simplicial complexes embedded in $\mathbb{R}^2$ or $\mathbb{R}^3$.*

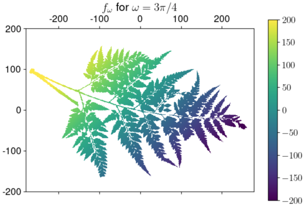
Generalizations by Ghrist et al; Curry et al

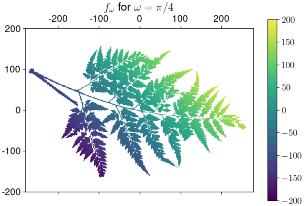
Section 3

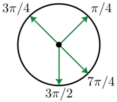
# The ECT

# Fern example

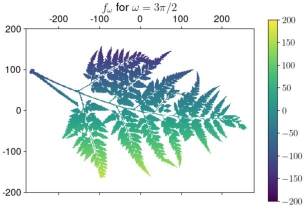
# Fern - Directional Transform

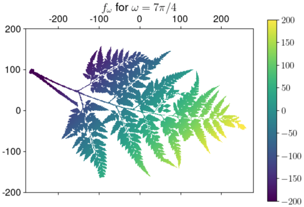


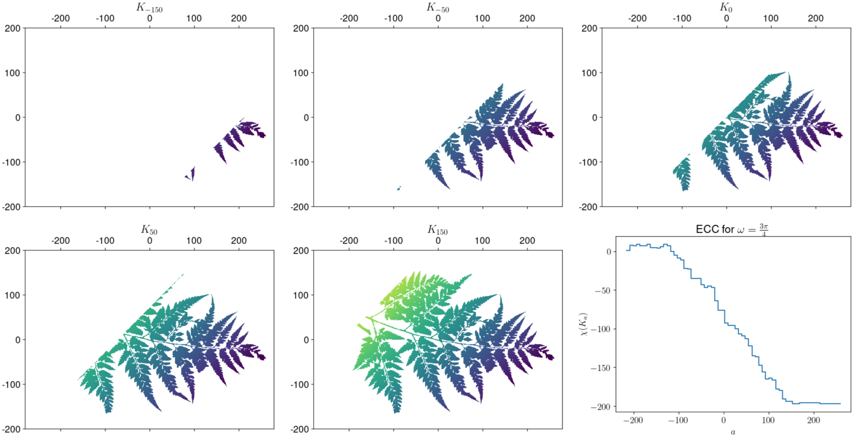




$$f_\omega(x) = \langle x, \omega \rangle$$



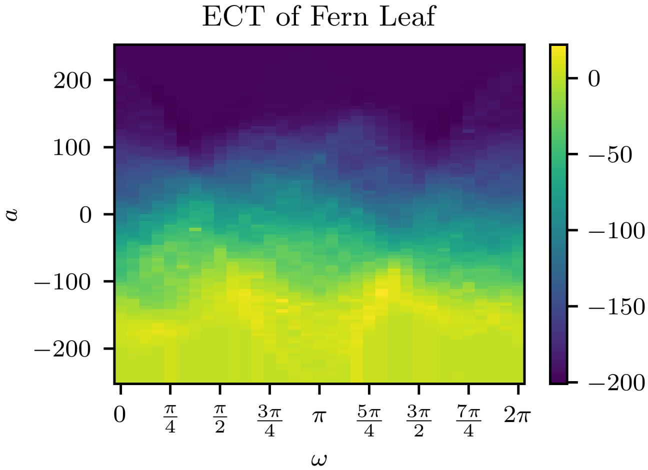
# Fern - Thresholds




$$ECC(a) = \chi(f_\omega^{-1}(-\infty, a])$$

# Fern - ECT

$$ECT(\omega, a) = \chi(f_\omega^{-1}(-\infty, a])$$



# Next time

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