

Topological Signal Processing

Lecture 24 - CMSE 890

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Michigan State University

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Dept of Computational Mathematics, Science & Engineering

Thurs, Dec 4, 2025

This lecture

- Time series analysis with persistence

Input: 1D Time series

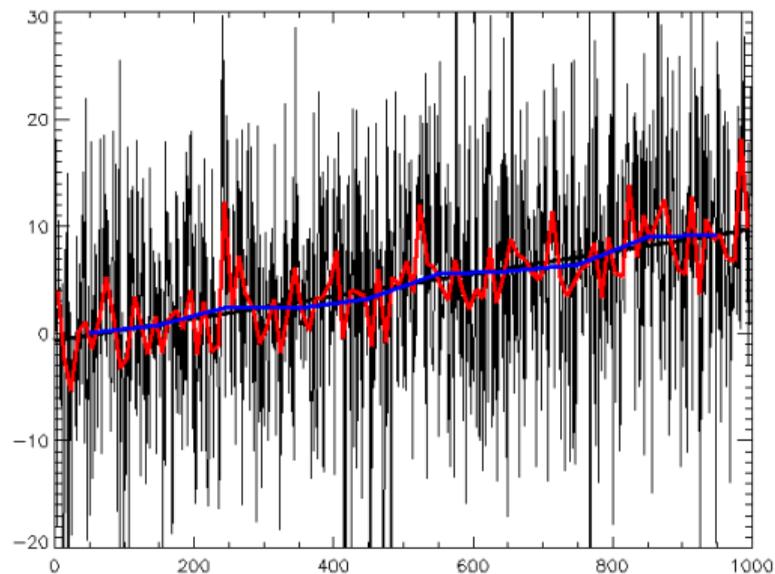
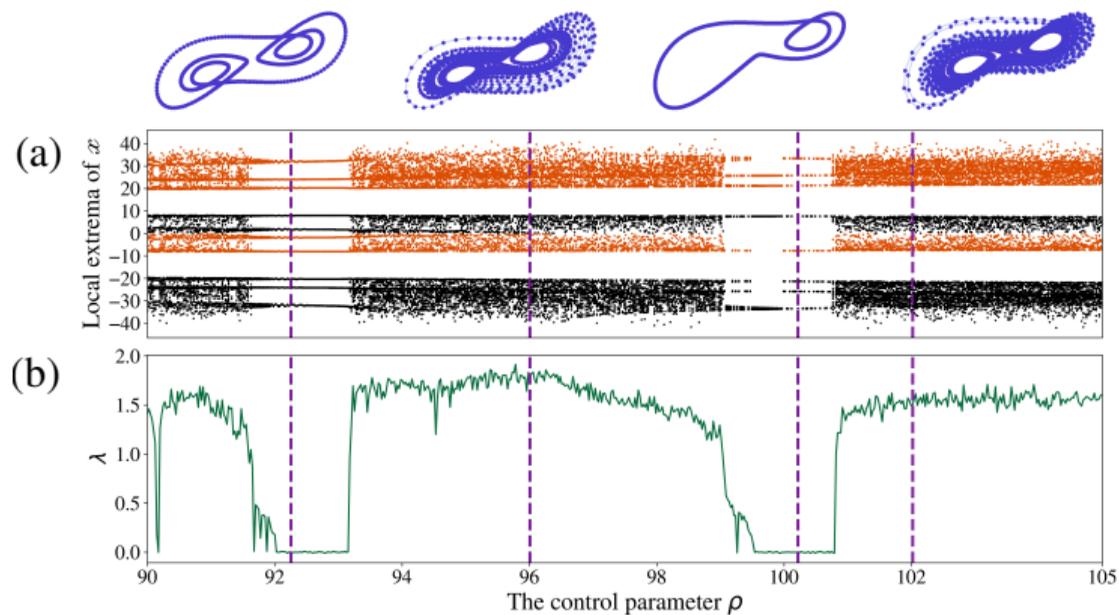


Image: Wikipedia

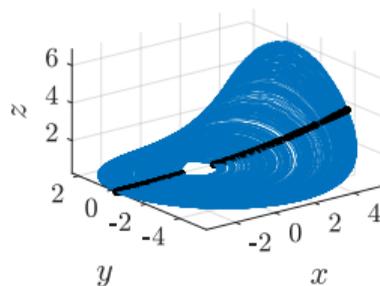
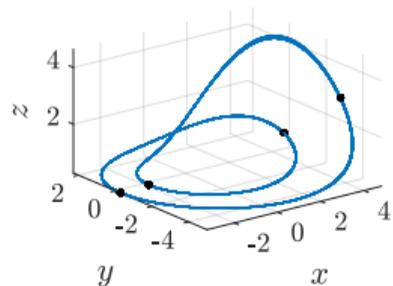
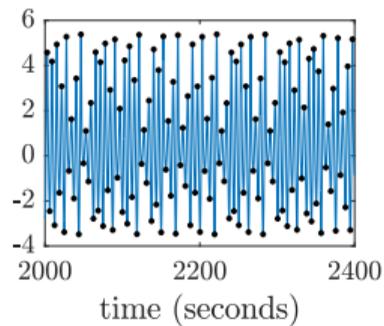
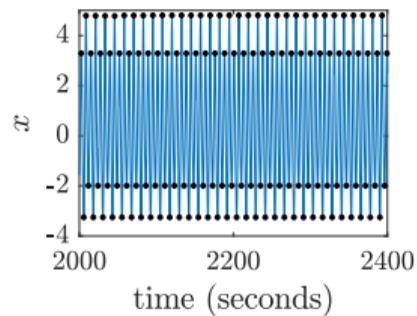
- $f : \mathbb{R} \rightarrow \mathbb{R}$
- Discrete samples $\{x_t\}_{t=1}^N$

Lorenz system



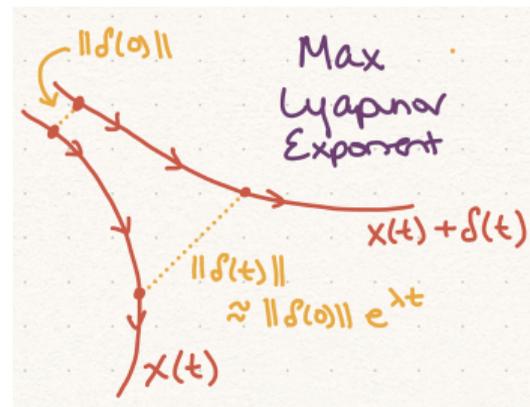
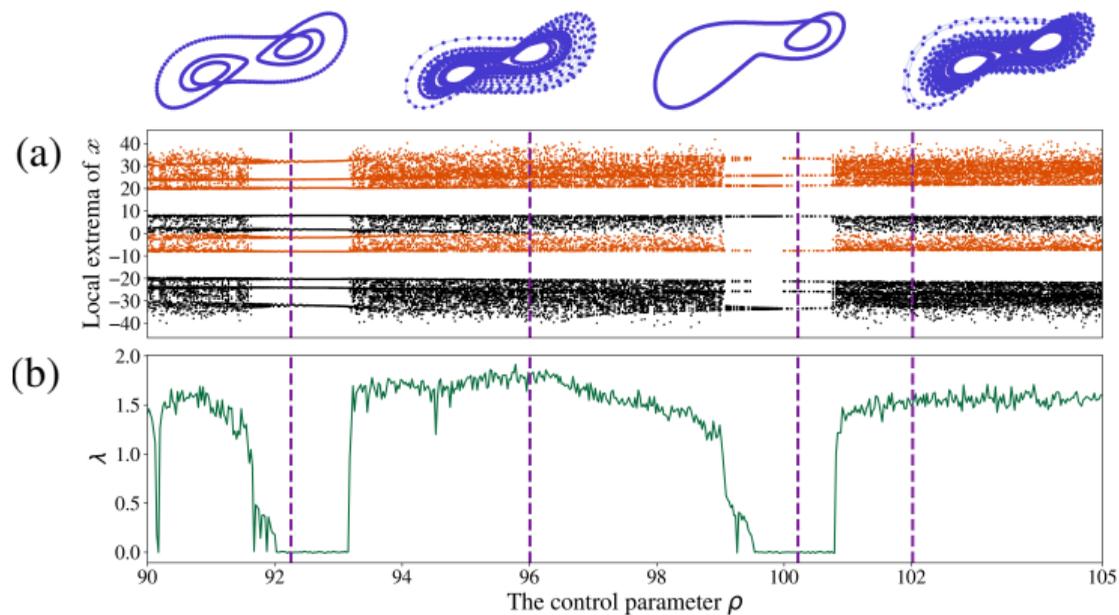
$$\begin{aligned}\dot{x} &= \sigma(y - x), \\ \dot{y} &= x(\rho - z) - y, \\ \dot{z} &= xy - \beta z,\end{aligned}$$

Rosler system

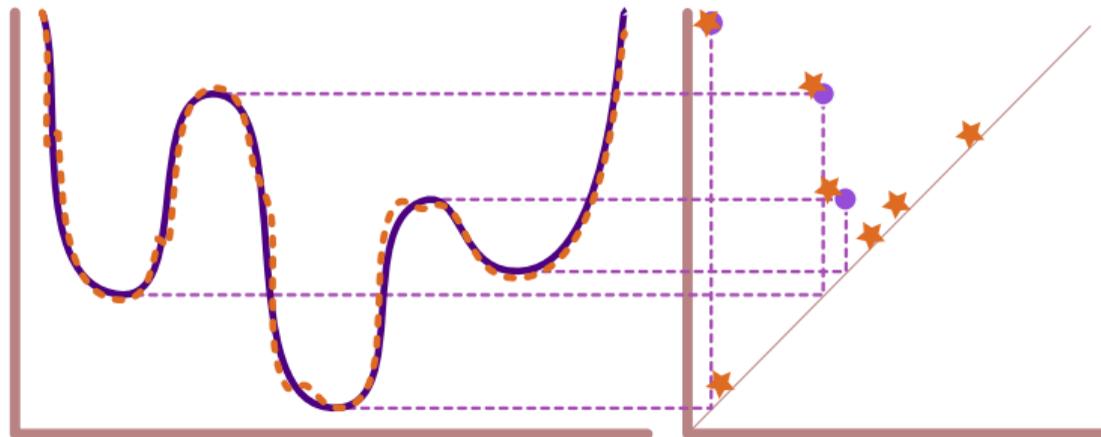


$$\begin{aligned}\dot{x} &= -y - z, \\ \dot{y} &= x + ay, \\ \dot{z} &= b + z(x - c),\end{aligned}$$

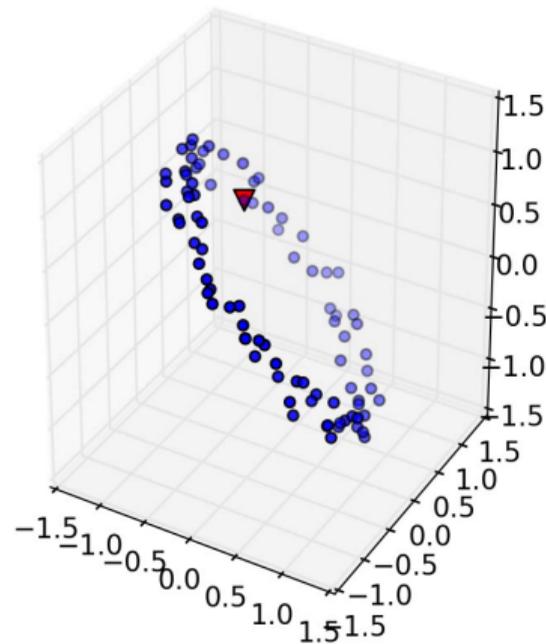
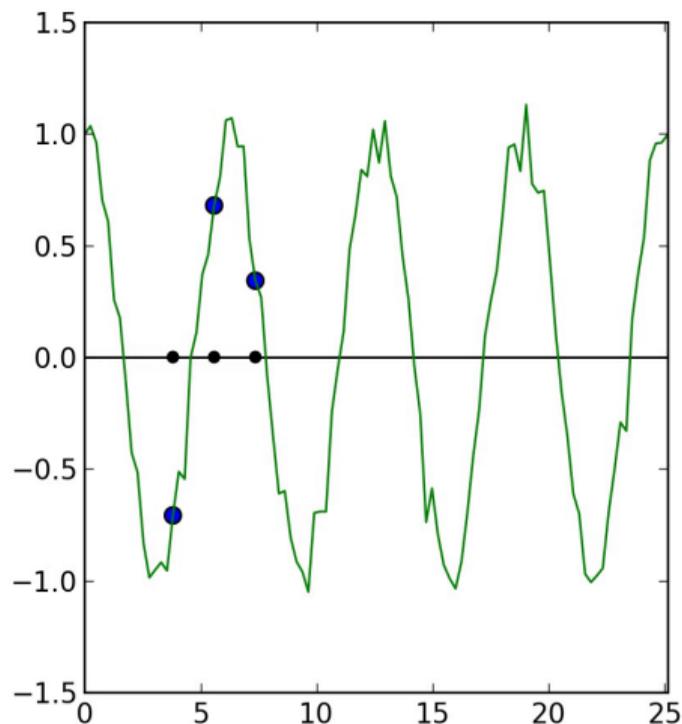
Maximum Lyapunov exponent



Option 0: Sublevelset persistence of time series



Takens' embedding



$$\Phi_{d,\tau} : [x(t_1), x(t_2), \dots, x(t_k)] \mapsto \{(x(t)), x(t + \tau), \dots, x(t + (d - 1)\tau)\}$$

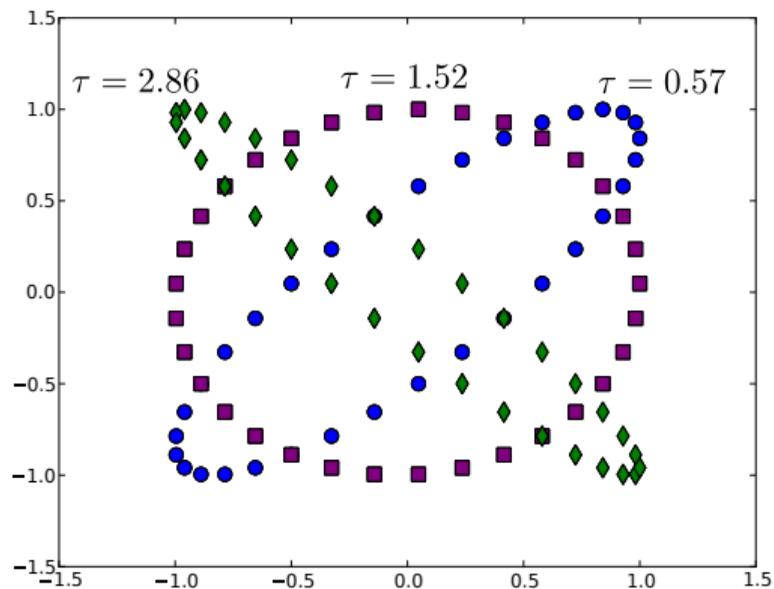
Takens Theorem

- Attractor A
- Given time series data from scalar measurements $s : A \rightarrow \mathbb{R}$
- Choose m, τ
- The delay reconstruction is $\{(s_{n-(m-1)\tau}, \dots, s_{n-\tau}, s_n)\}_n \subseteq \mathbb{R}^m$

Theorem (Takens 1981; Sauer, Yorke, Casdagli 1991)

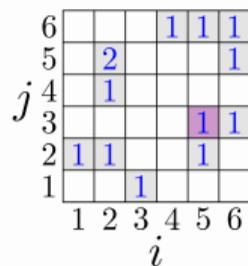
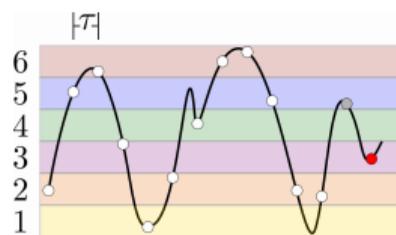
For generic smooth measurement functions s , the map from the attractor A to the delay reconstruction in \mathbb{R}^m is an embedding when $m > 2\dim(A)$ where $\dim(A)$ is the box counting dimension.

Example with different delays



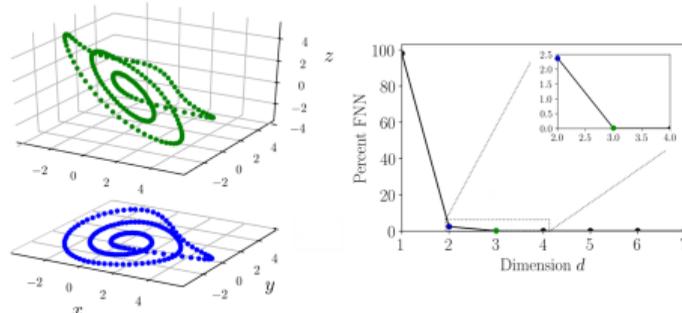
Mutual Information

$$I_\epsilon(\tau) = \sum_{i,j} p_{ij}(\tau) \ln p_{ij}(\tau) - 2 \sum_i p_i \ln p_i,$$



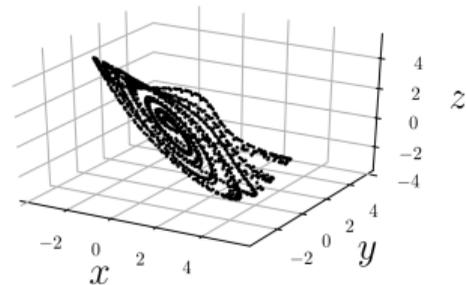
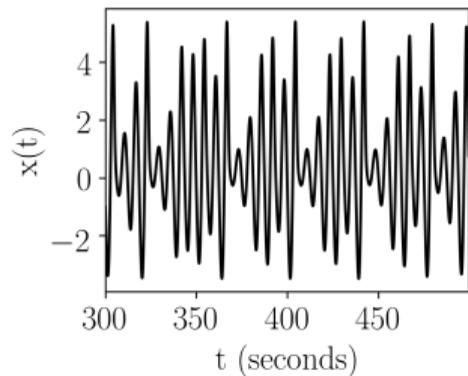
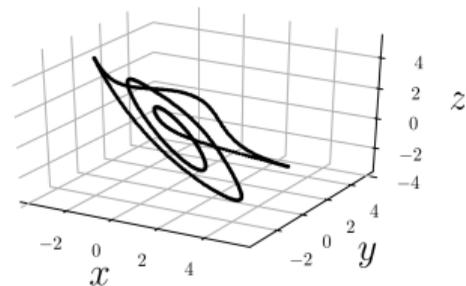
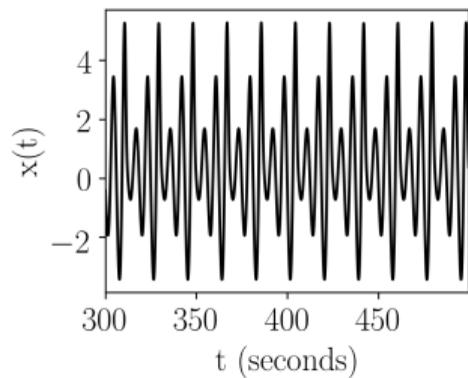
Choose τ to be first minimum of $I_\epsilon(\tau)$

False Nearest Neighbors

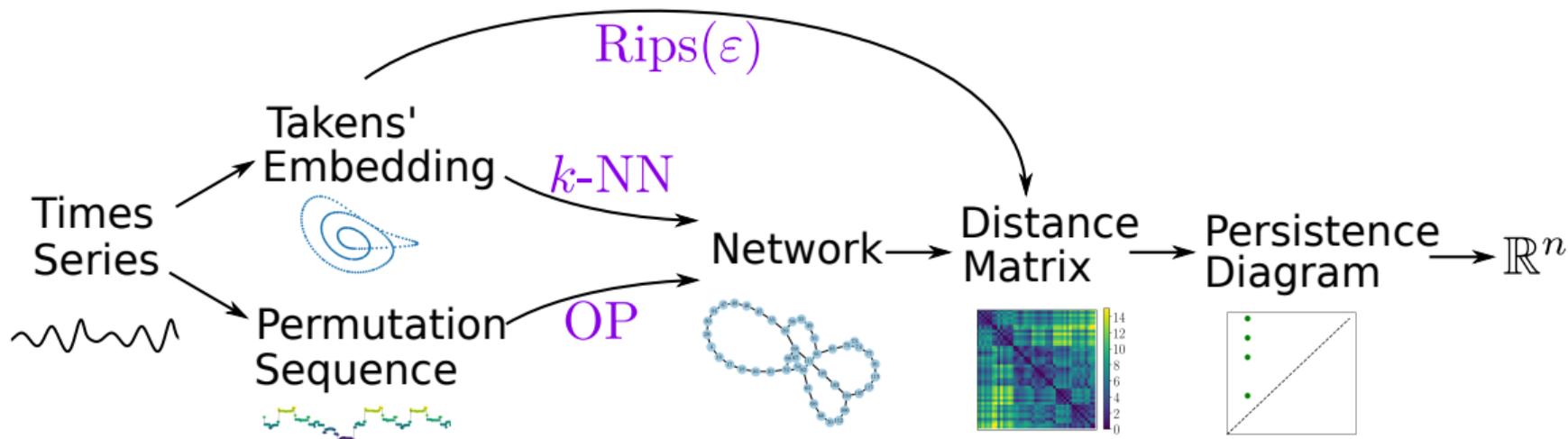


Choose d to be smallest when FNN is below some threshold.

Distinguish periodic vs chaotic



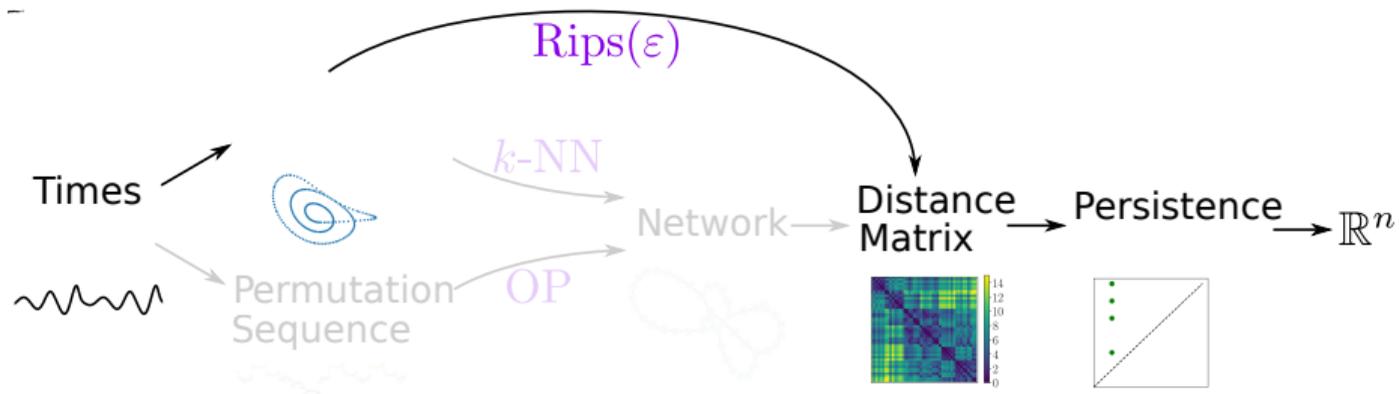
From Time Series to Scores



Section 1

Rips Complex from Takens Embedding

Pipeline

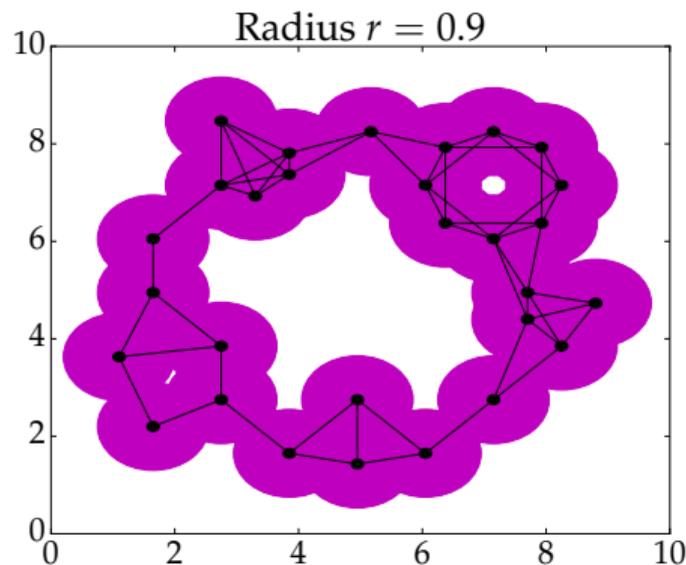


Persistence on point clouds: Rips complex

Definition

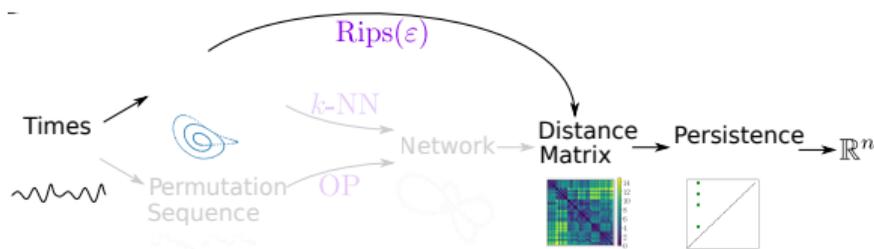
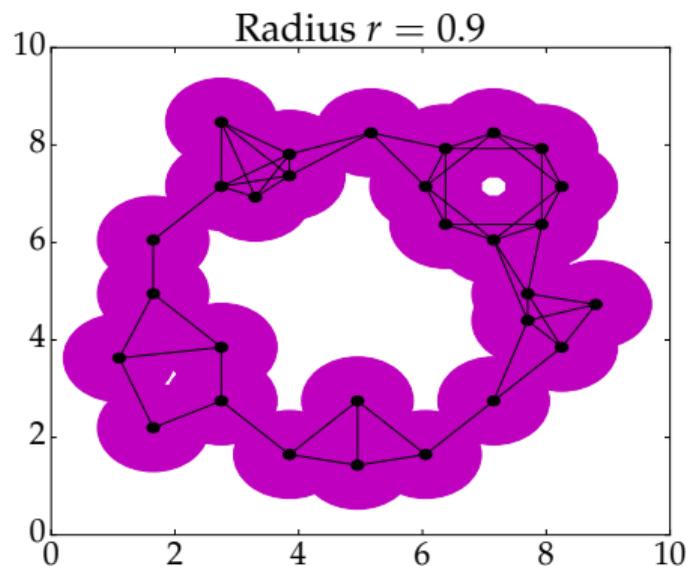
Given a point cloud $\chi \subseteq \mathbb{R}^d$,

$$\text{Rips}(\varepsilon) = \{\sigma \subseteq \chi \mid \|u - v\| \leq \varepsilon \forall u, v \in \sigma\}$$

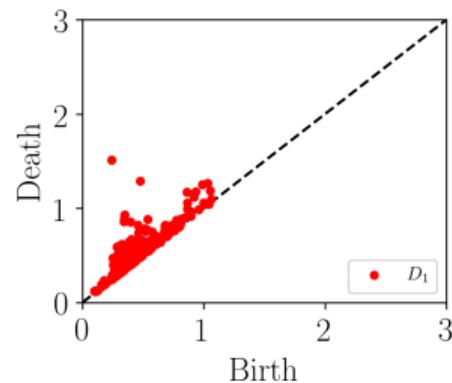
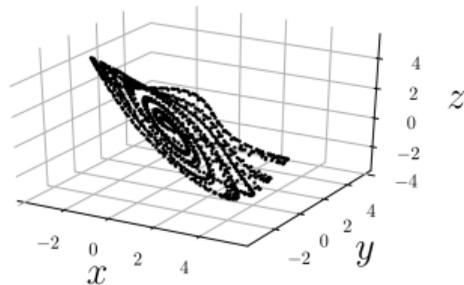
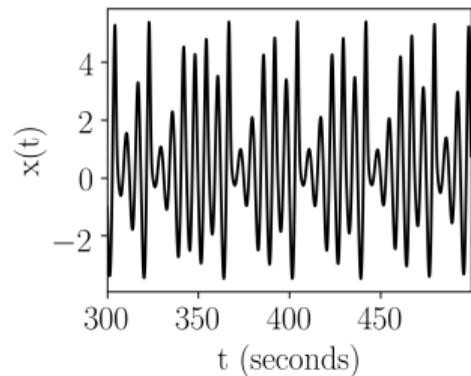
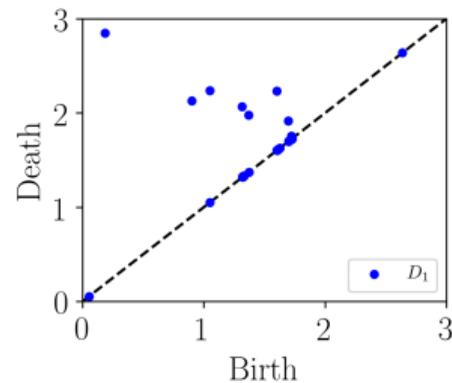
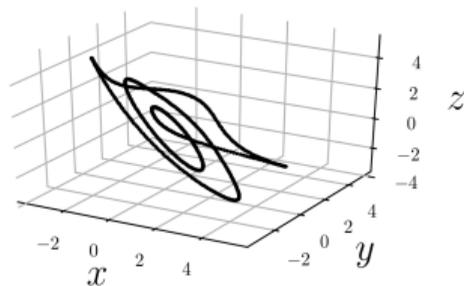
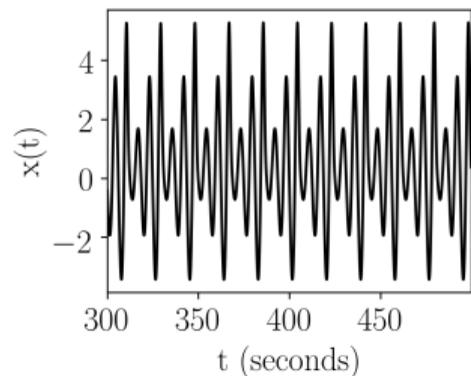


Method 1: Rips complex of Takens' embedding

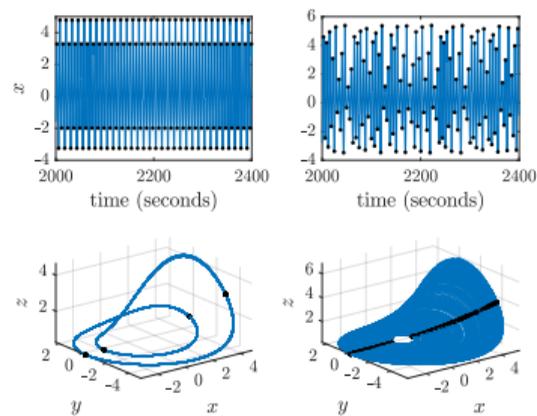
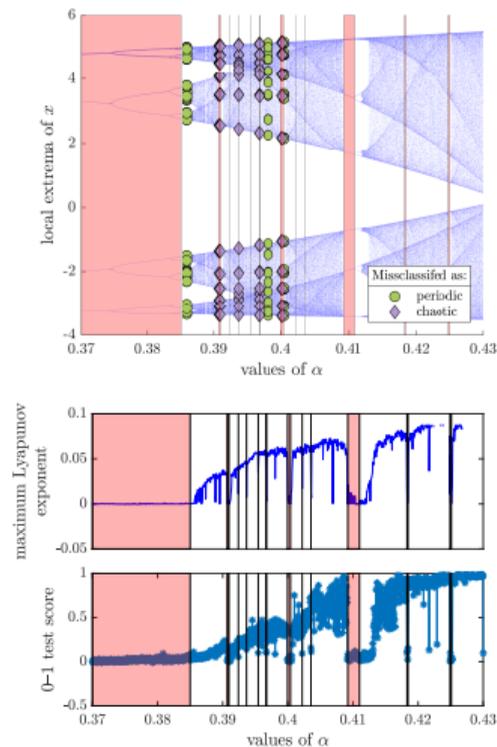
- Fix parameters d, τ
- Construct point cloud
- Compute distance matrix in \mathbb{R}^d
- Compute persistence diagram



Example of Rips complex pipeline on Rossler



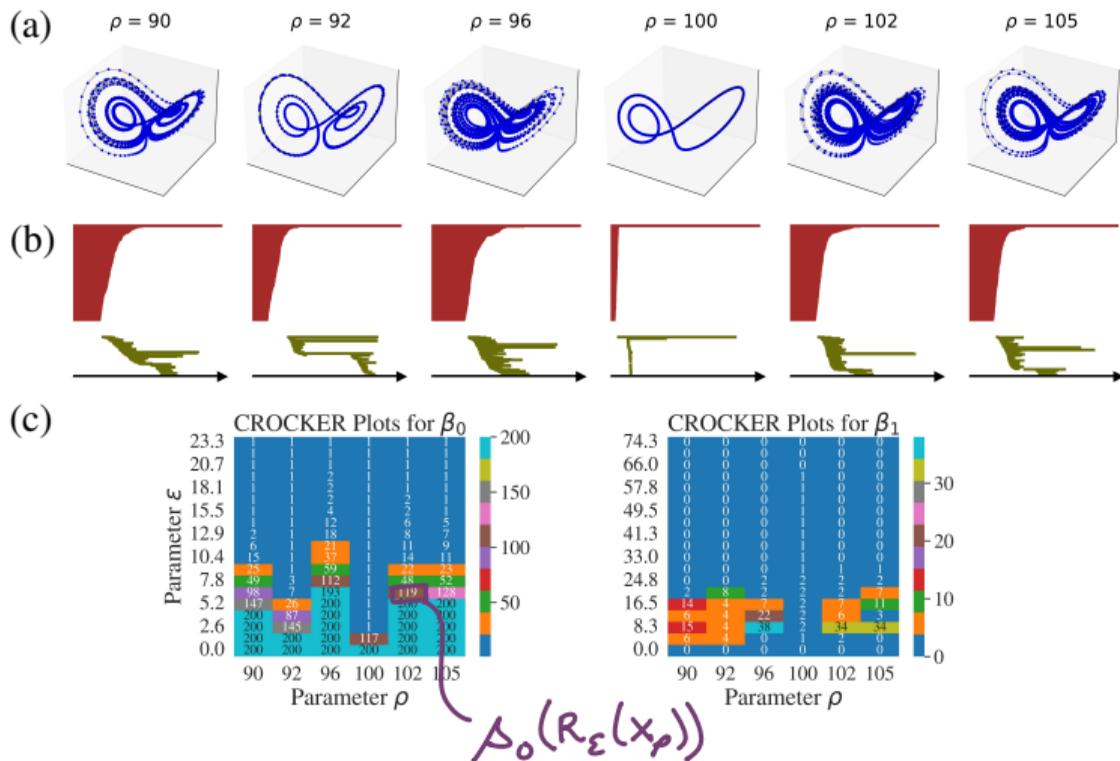
Rosler classification with tent functions



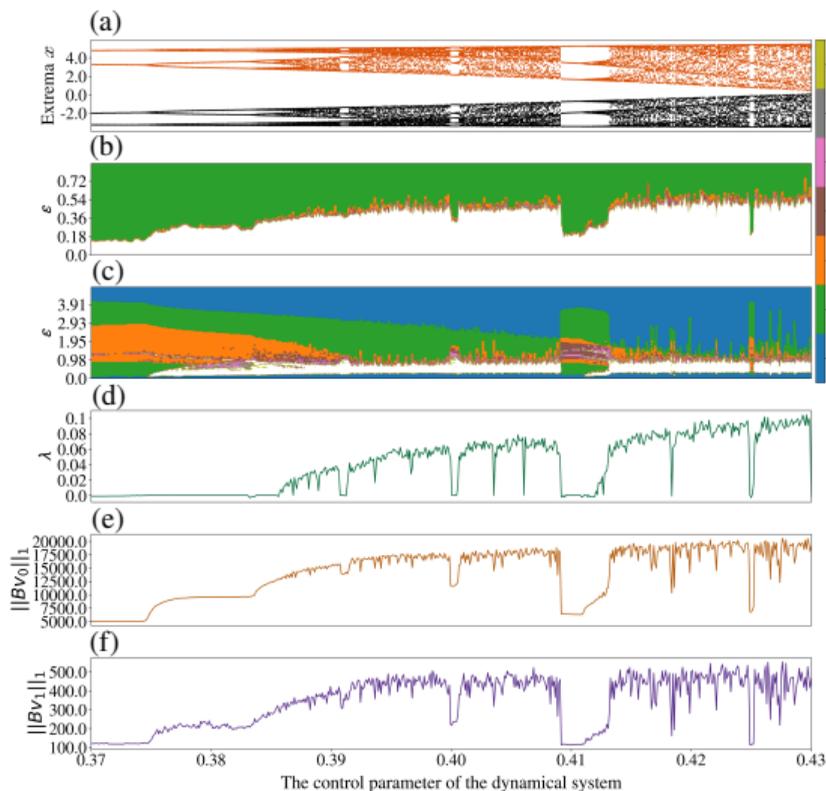
- 1 Use Lyapunov to decide on ground truth
- 2 Supervised learning on the template features
- 3 Misclassified labeled with circles/diamonds

Perea, Munch, Khasawneh. *Approximating Continuous Functions on Persistence Diagrams Using Template Functions*, arXiv:1902.07190, 2019.

Recall: CROCKER



Crocker: Lorenz



2 Bifurcation diagram

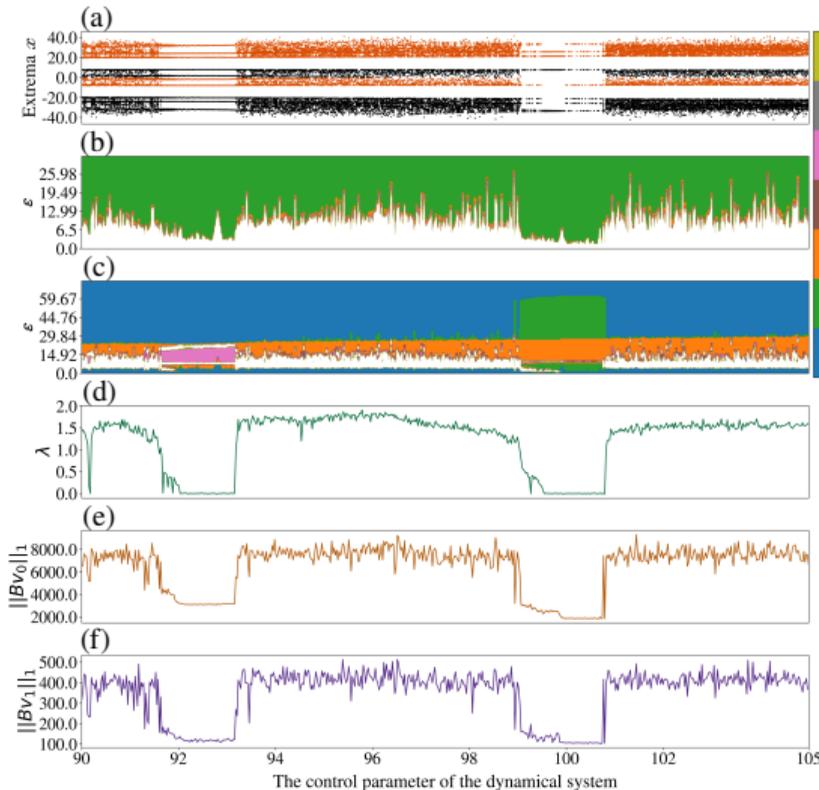
— 0-dim Crocker

— 1-dim Crocker

— Lyapunov exponent

Detecting bifurcations in dynamical systems with CROCKER plots. İsmail Güzel, Elizabeth Munch, and Firas A. Khasawneh. Chaos 2022.

Crocker: Rössler



- Bif

} Crocker

- Lyapunov

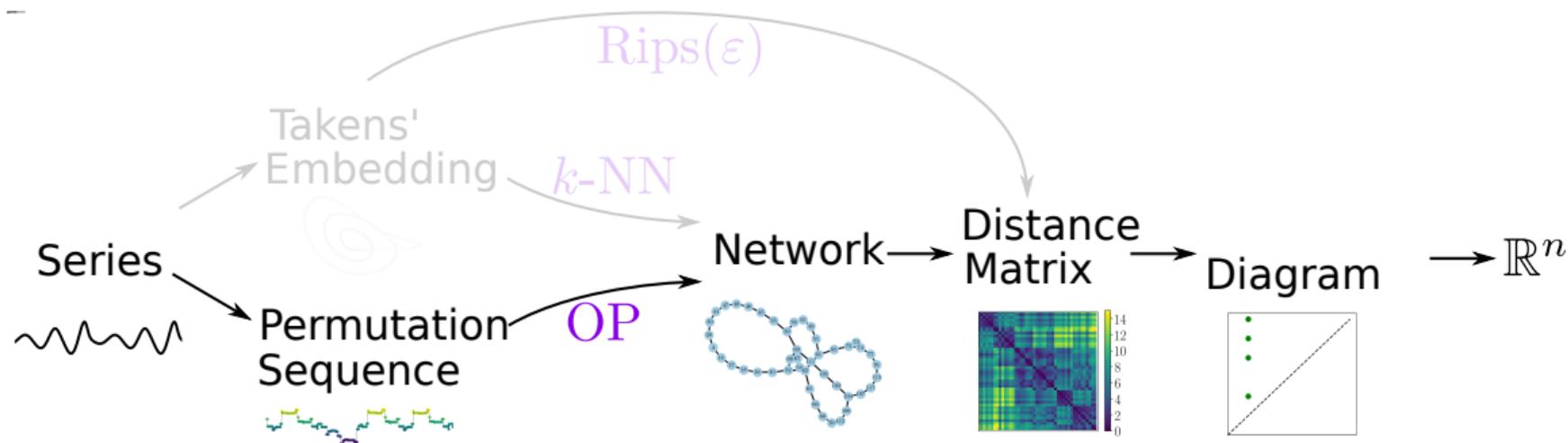
} norms of Crocker

Detecting bifurcations in dynamical systems with CROCKER plots. İsmail Güzel, Elizabeth Munch, and Firas A. Khasawneh. Chaos 2022.

Section 2

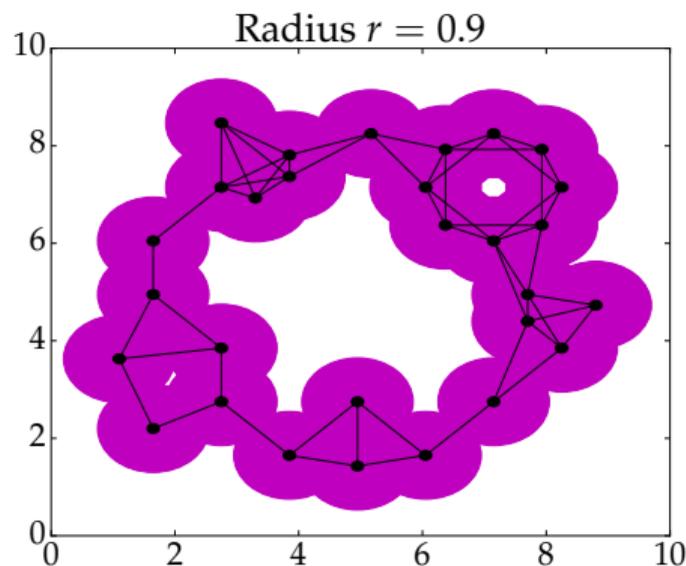
Graph-based methods from Takens Embedding

Second pipeline

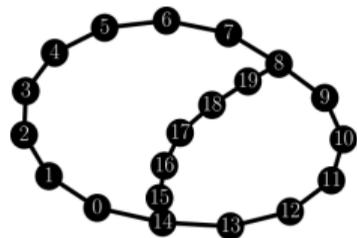


Inducing a filtration from a distance

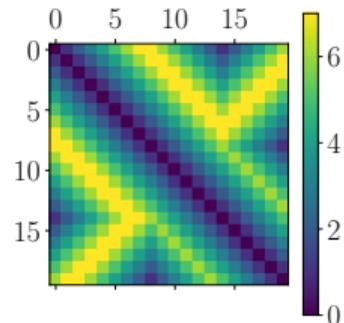
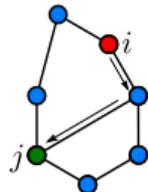
- 1 Given a distance $d : V \times V \rightarrow \mathbb{R}_{\geq 0}$.
- 2 K is complete simplicial complex with vertices V
- 3 $f : K \rightarrow \mathbb{R}_{\geq 0}$ given by
$$\sigma \mapsto \max_{\{u,v\} \subseteq \sigma} \{d(u,v)\}$$
- 4 $K_a = \{\sigma \in K \mid f(\sigma) \leq a\}$
- 5 Filtration $K_{a_1} \subseteq K_{a_2} \subseteq \dots \subseteq K_{a_k}$



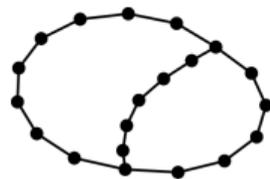
Persistence on graph



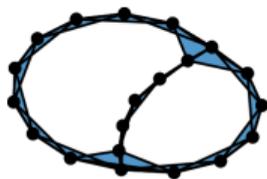
$d(u, v) = \text{shortest path length}$



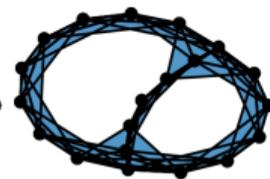
K_0



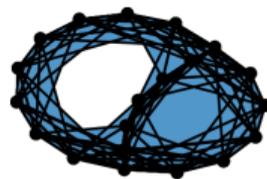
K_1



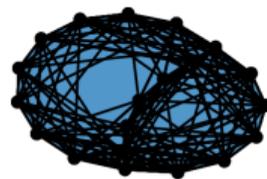
K_2



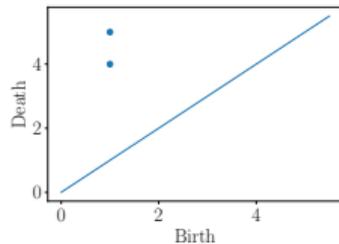
K_3



K_4



K_5

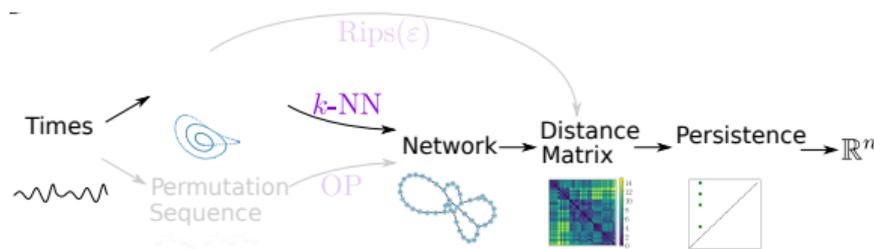


Method 2: k -NN graph of Takens' embedding

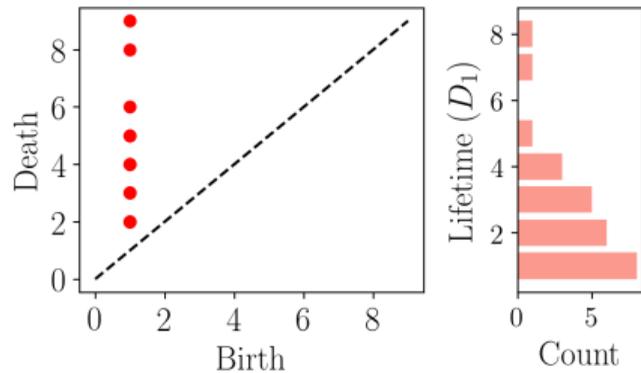
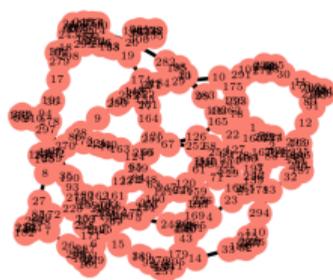
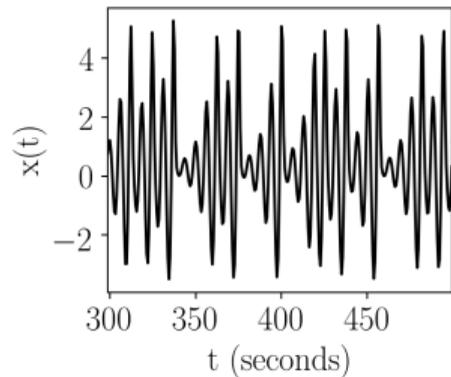
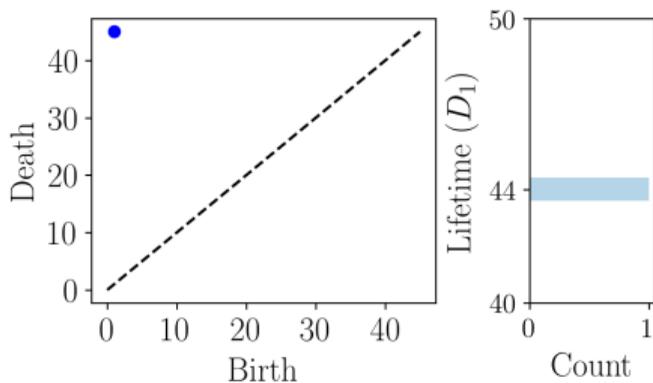
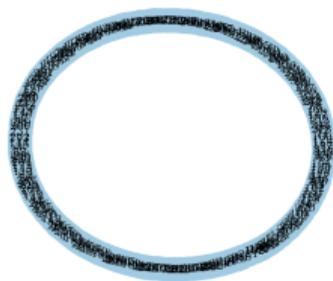
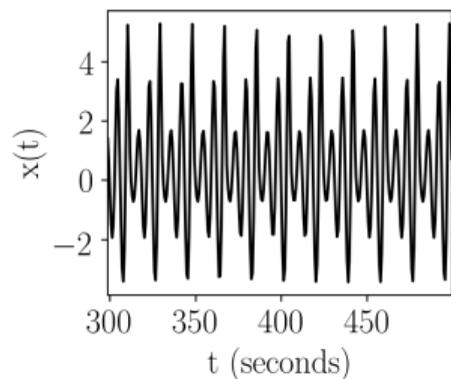
Definition

The (undirected, unweighted) k -NN graph of a point cloud χ has vertex set $V \cong \chi$ and an edge uv if $v \in \chi$ is among the k closest neighbors of v .

- Fix parameters d, τ
- Construct point cloud
- Choose k
- Construct k -NN Graph
- Compute shortest path distance in the graph
- Compute persistence diagram



k -NN graph of Takens' embedding example



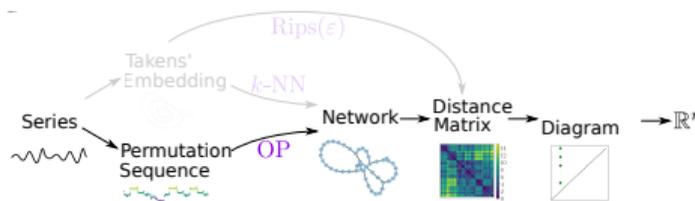
Method 2: Ordinal partition network

Small 2013; McCullough, Small, Stemler, Iu 2015

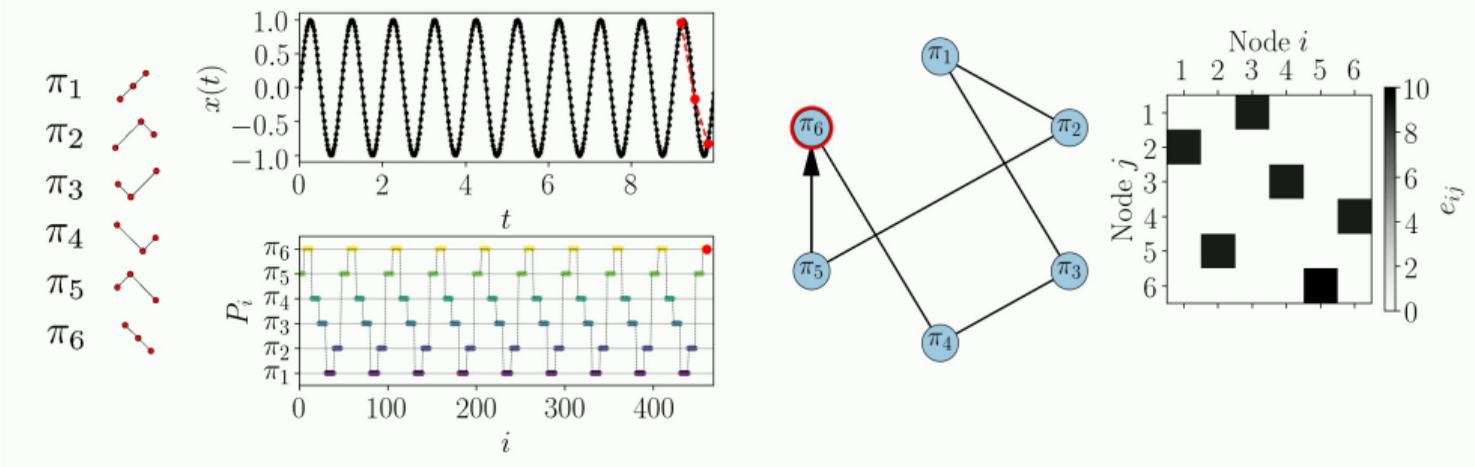
- Fix parameters d, τ
- Construct point cloud
- Construct ordinal partition network
- Compute shortest path distance in the graph
- Compute persistence diagram

Definition

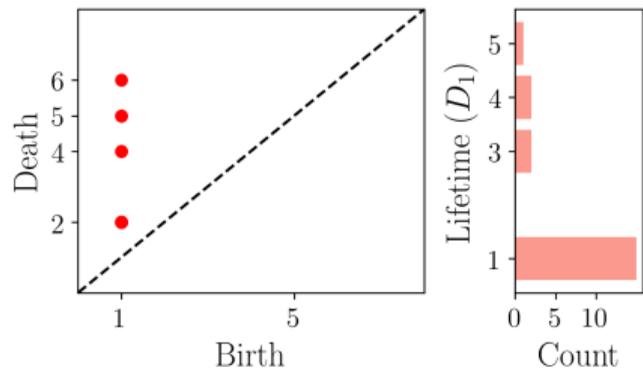
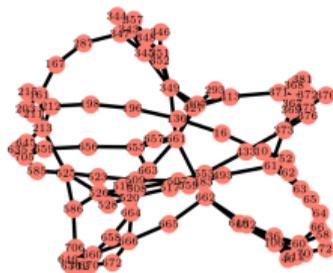
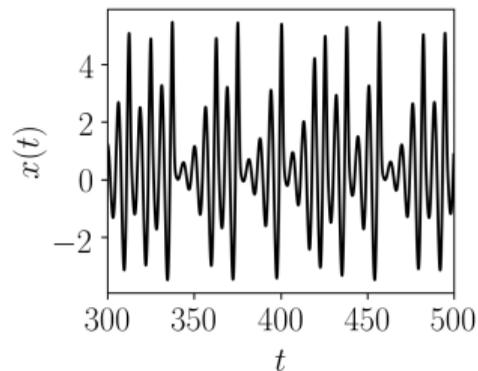
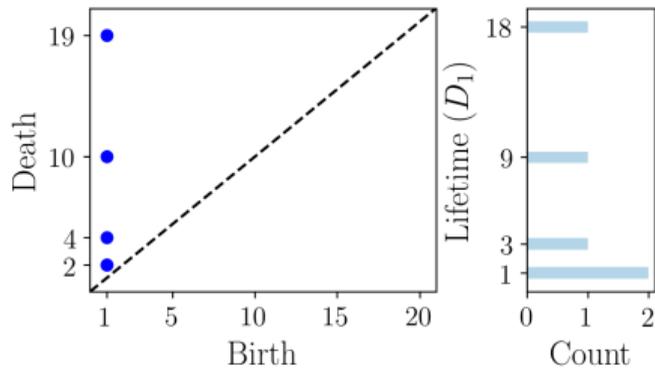
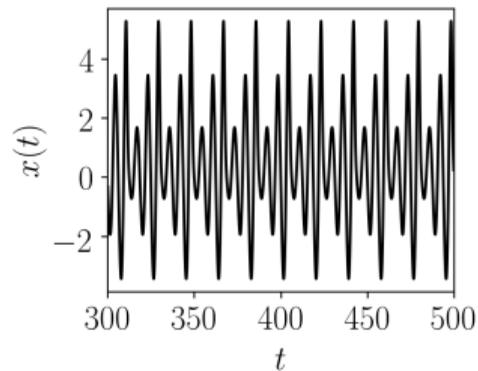
Given a point in $\chi, (x_1, \dots, x_d)$, the associated permutation π is the permutation of the set $\{1, \dots, d\}$ that satisfies $x_{\pi(1)} \leq x_{\pi(2)} \leq \dots \leq x_{\pi(d)}$.



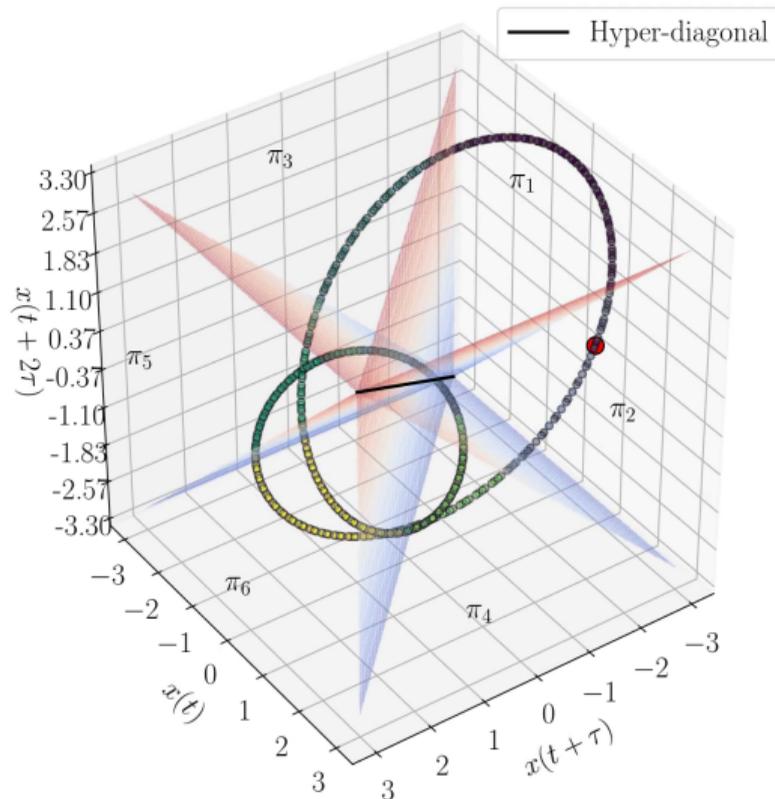
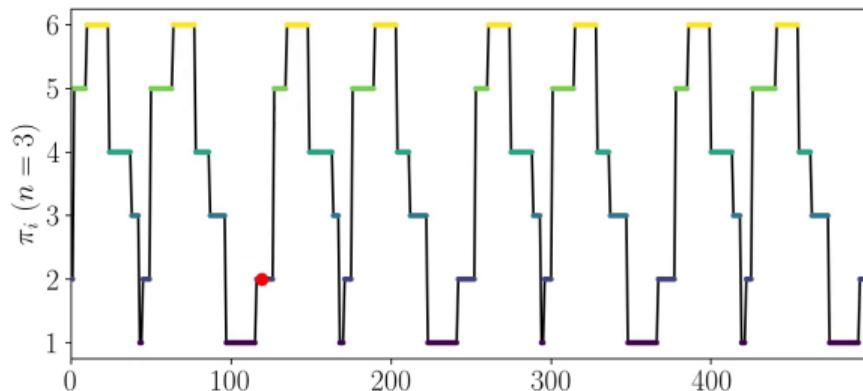
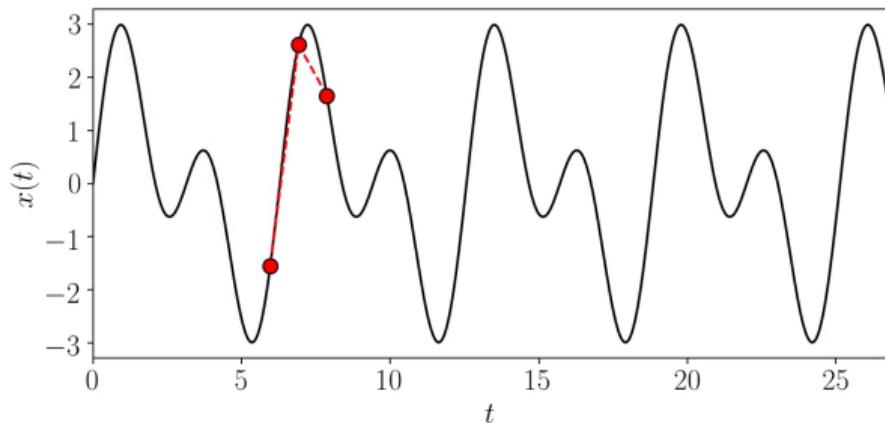
Ordinal Partition Network Example



Ordinal partition network of Rossler examples



Relationship to Takens Embedding



Definition (Chintakunta et al. 2015)

The persistent entropy of a diagram D is

$$E(D) = - \sum_{x \in D} \frac{\text{pers}(x)}{\mathcal{L}(D)} \log_2 \left(\frac{\text{pers}(x)}{\mathcal{L}(D)} \right),$$

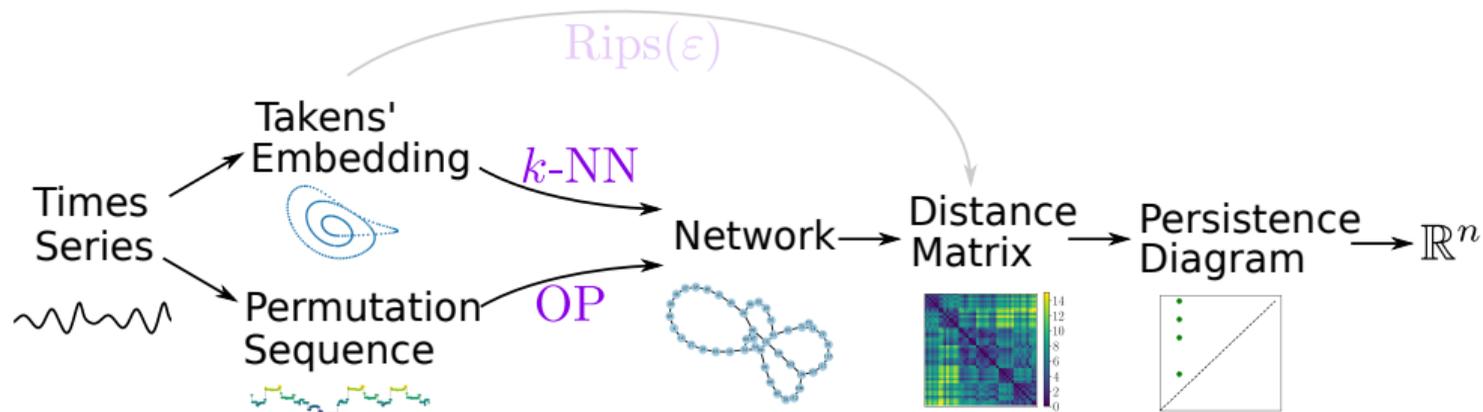
where $\text{pers}(x_1, x_2) = |x_2 - x_1|$ and $\mathcal{L}(D) = \sum_{x \in D} \text{pers}(x)$.

Definition

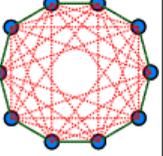
The normalized persistent entropy of a diagram D is

$$E'(D) = \frac{E(D)}{\log_2(\mathcal{L}(D))}.$$

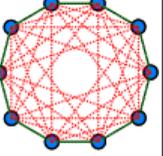
Pipeline



Periodicity score

n	Cycle	r_B	r_D	L_n
3		1	1	0
4		1	2	1
5		1	2	1
6		1	2	1
	⋮			
10		1	4	3
	⋮			

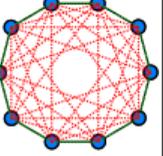
Periodicity score

n	Cycle	r_B	r_D	L_n
3		1	1	0
4		1	2	1
5		1	2	1
6		1	2	1
	⋮			
10		1	4	3
	⋮			

Persistence diagram has 1
point at $(1, \lceil \frac{n}{3} \rceil)$.
Lifetime is

$$L_n = \text{maxpers}(D') = \lceil \frac{n}{3} \rceil - 1.$$

Periodicity score

n	Cycle	r_B	r_D	L_n
3		1	1	0
4		1	2	1
5		1	2	1
6		1	2	1
⋮	⋮			
10		1	4	3
⋮	⋮			

Persistence diagram has 1 point at $(1, \lceil \frac{n}{3} \rceil)$.
Lifetime is

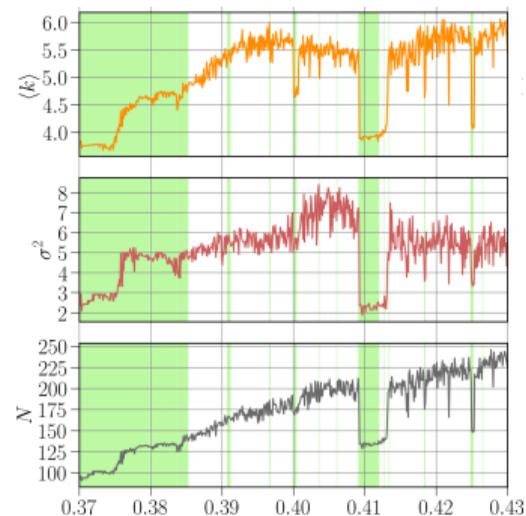
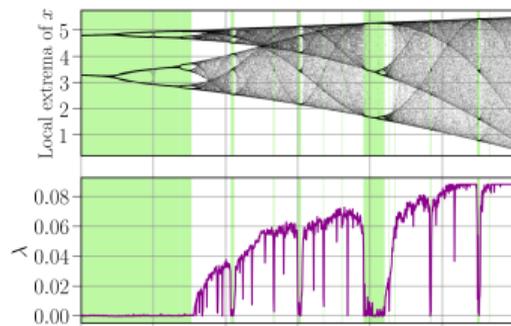
$$L_n = \text{maxpers}(D') = \lceil \frac{n}{3} \rceil - 1.$$

Definition

The network periodicity score is

$$P(D) = 1 - \frac{\text{maxpers}(D)}{L_n}.$$

Ordinal Partition Network Results



$P(D)$: periodicity score

$E'(D)$: normalized persistent entropy

$\langle k \rangle$: mean out degree

σ^2 : variance of out degree

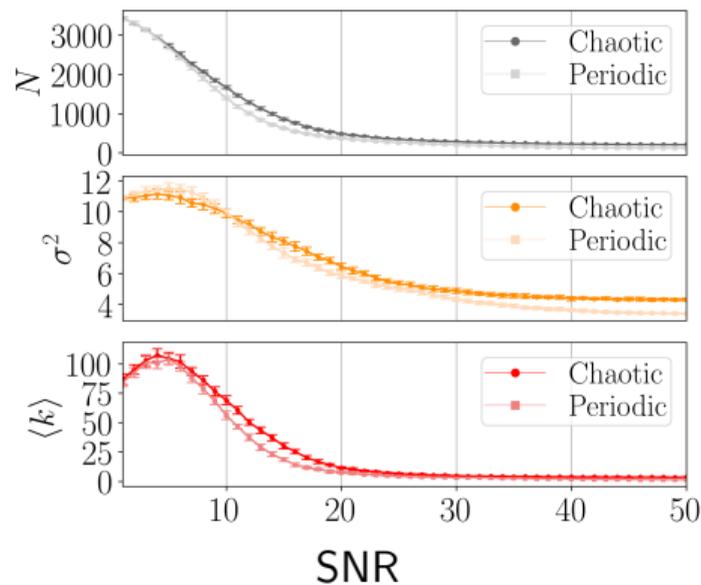
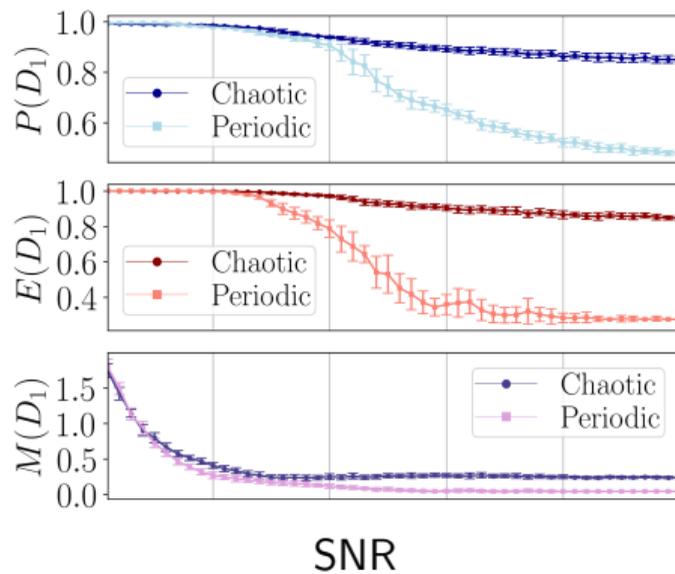
N : number of nodes

Results on other systems

System/ Data	k -NN Graph from Takens' Embedding						Ordinal Partition Graph					
	$E'(D_1)$		$M(D_1)$		$P(D_1)$		$E'(D_1)$		$M(D_1)$		$P(D_1)$	
	Per.	Ch.	Per.	Ch.	Per.	Ch.	Per.	Ch.	Per.	Ch.	Per.	Ch.
Chua Circuit	0.00	0.80	0.001	0.19	0.54	0.89	0.21	0.72	0.051	0.19	0.42	0.88
Lorenz	0.04	0.84	0.005	0.16	0.64	0.93	0.18	0.95	0.026	0.36	0.28	0.96
Rossler	0.00	0.85	0.001	0.18	0.50	0.94	0.00	0.89	0.036	0.28	0.33	0.85
Coupled Lorenz-Rossler	0.00	0.82	0.003	0.16	0.46	0.94	0.00	0.87	0.033	0.35	0.56	0.92
Bi-directional Rossler	0.00	0.76	0.004	0.13	0.55	0.87	0.25	0.91	0.064	0.29	0.40	0.92
Mackey-Glass	0.00	0.67	0.001	0.07	0.56	0.93	0.30	0.96	0.077	0.37	0.25	0.93

Myers, Munch, Khasawneh. *Persistent Homology of Complex Networks for Dynamic State Detection*, Phys. Rev. E, 2019.

Resilience to additive noise



Myers, Munch, Khasawneh. *Persistent Homology of Complex Networks for Dynamic State Detection*, Phys. Rev. E, 2019.

Section 3

Zigzags and the BuZZ method

Zigzag persistent homology

- Given a collection of simplicial complexes with inclusions **that go in different directions**,

$$K_0 \hookrightarrow K_1 \leftarrow K_2 \hookrightarrow \cdots \leftarrow K_n$$

then the inclusion maps induce maps on the p -th homology groups **that go in different directions**,

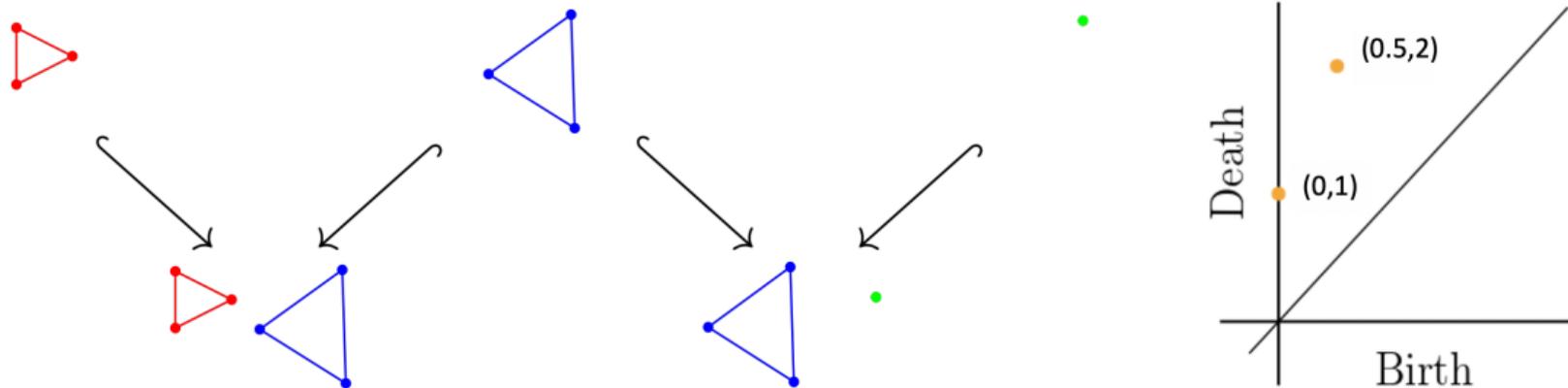
$$H_p(K_0) \rightarrow H_p(K_1) \leftarrow H_p(K_2) \rightarrow \cdots \leftarrow H_p(K_n).$$

- This a sequence of vector spaces and linear maps called a **zigzag module**.

Zigzag persistence

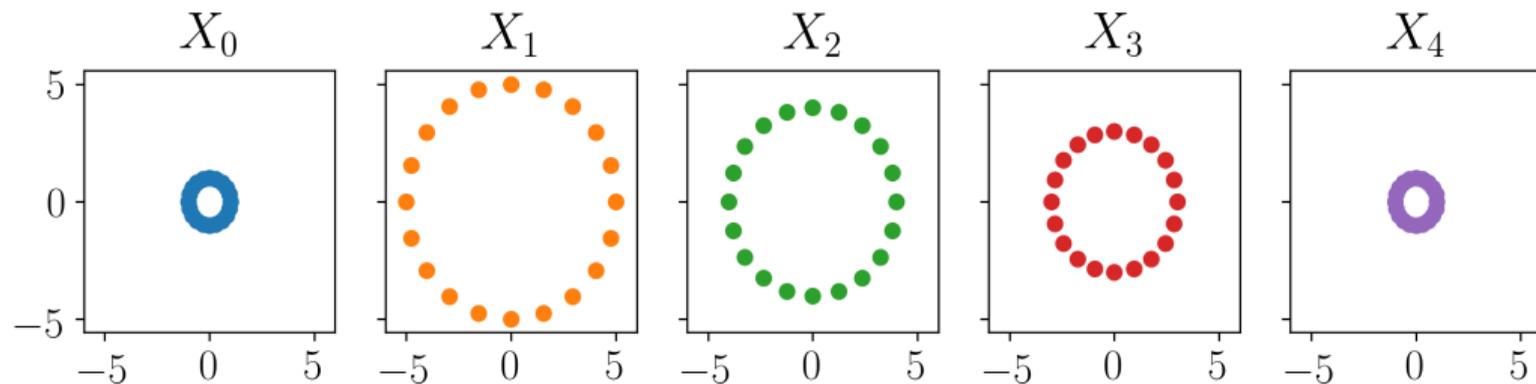
- For a zigzag module \mathbb{V} , the **zigzag persistence** of \mathbb{V} is the multiset of birth, death pairs

$$\text{Pers}(\mathbb{V}) = \{[b_j, d_j] \subset \{1, \dots, n\} \mid j = 1, \dots, N\}.$$

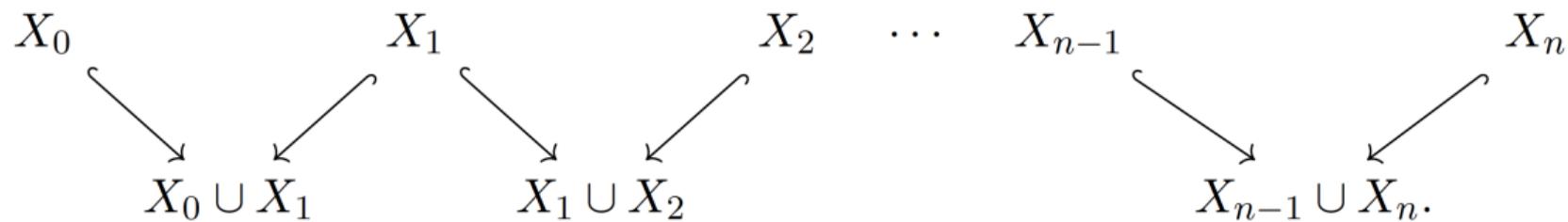


Motivation

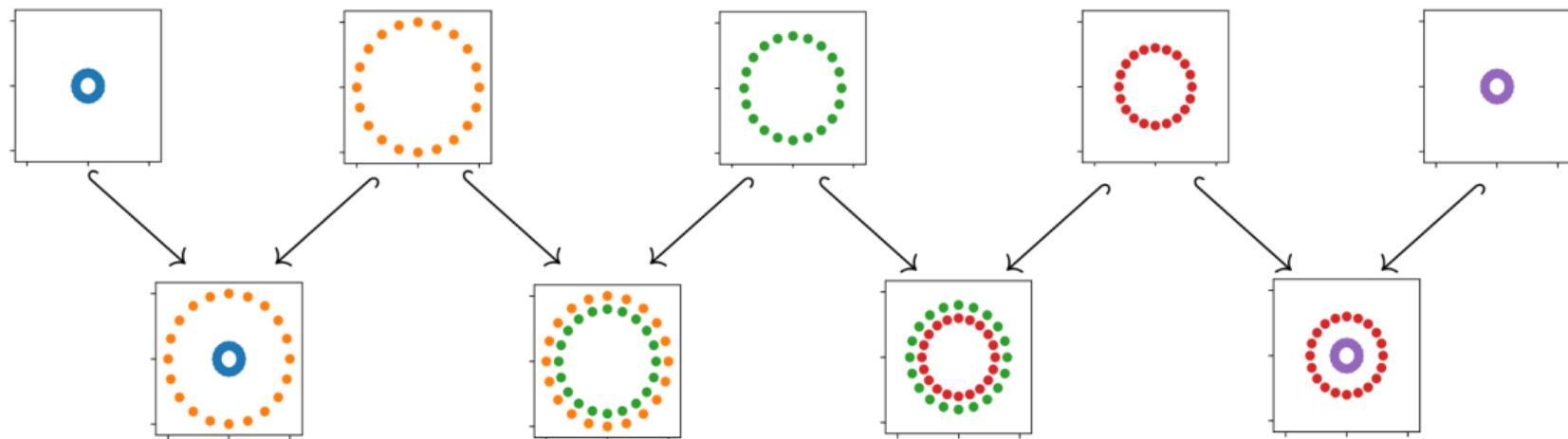
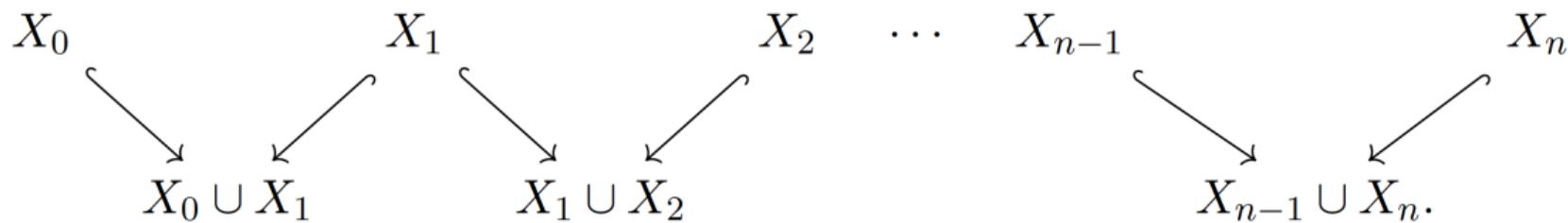
Using zigzag persistence, can we detect the range of point clouds that are “most circular”?



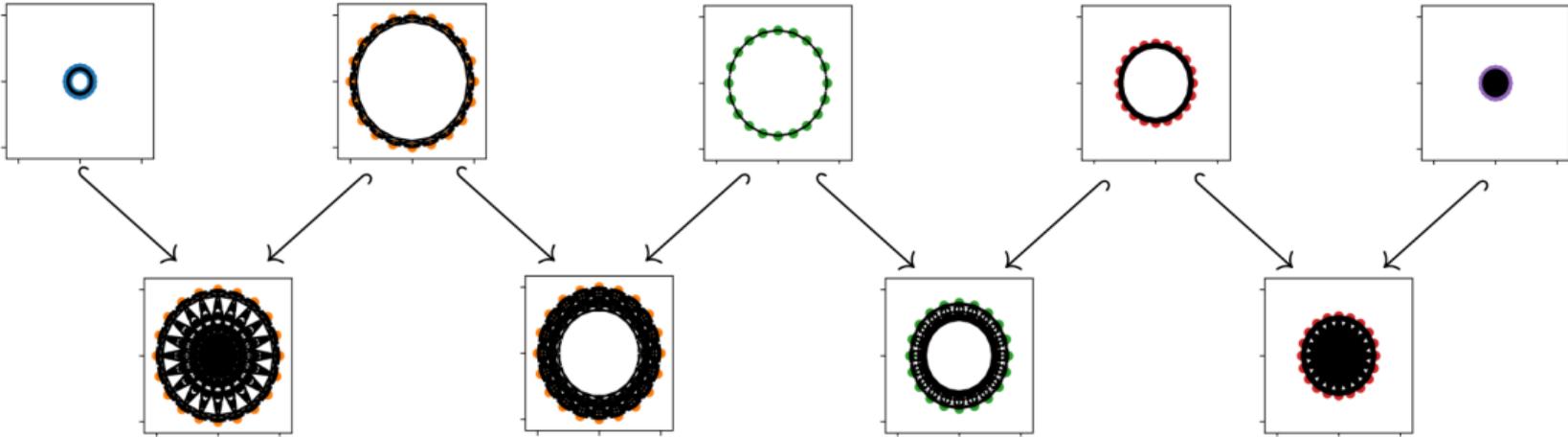
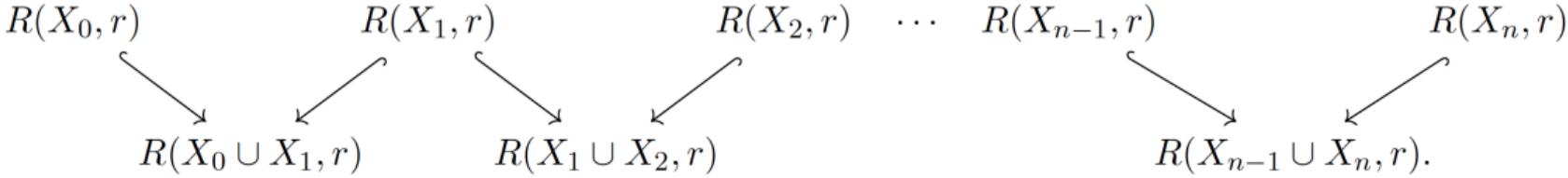
Setting up the zigzag



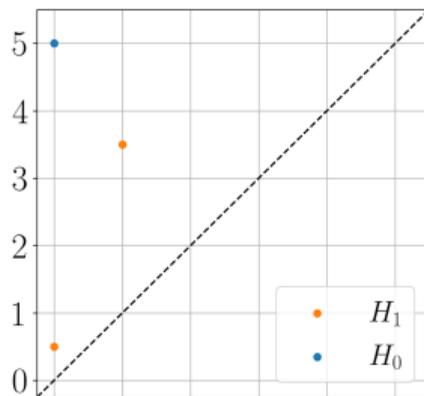
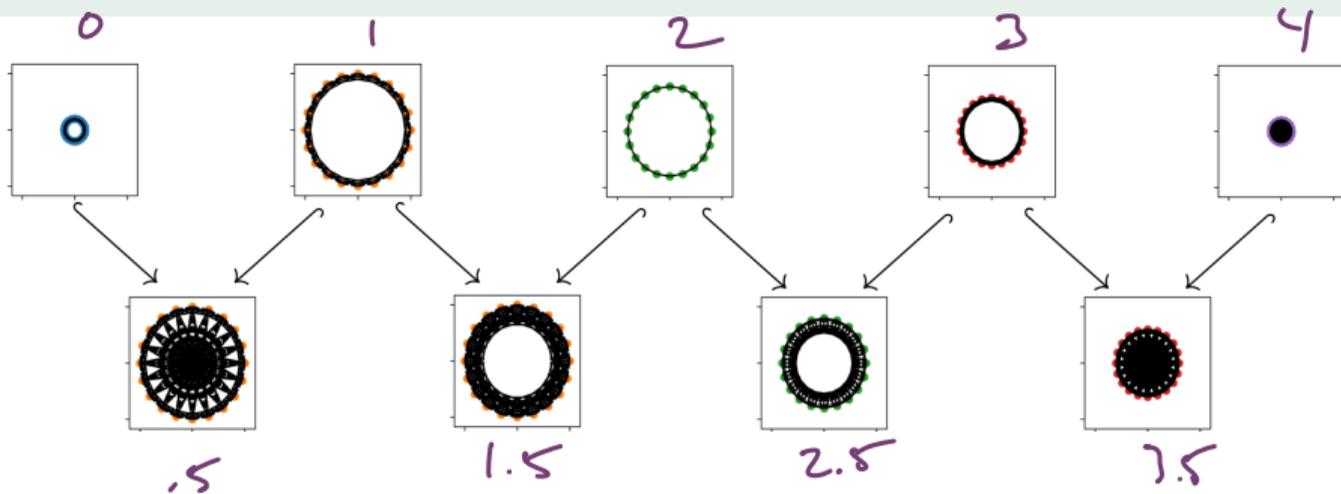
Setting up the zigzag



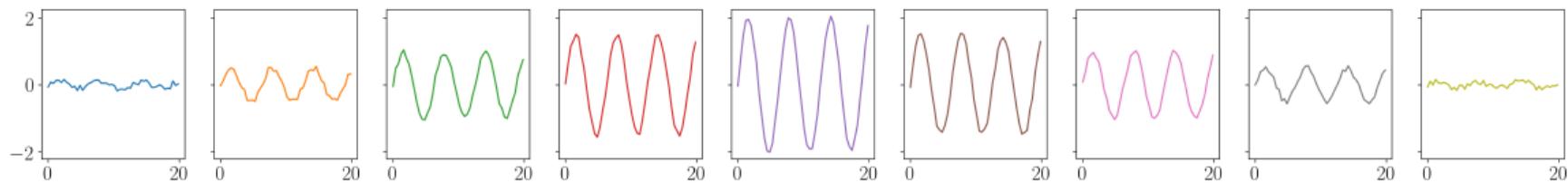
Zigzag diagram of Rips complexes



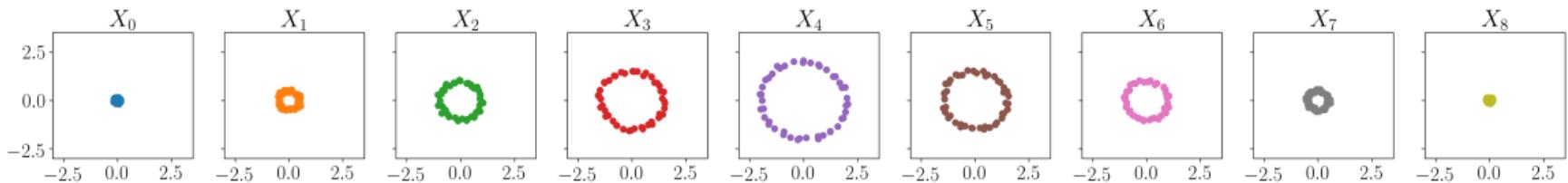
How to interpret the diagrams



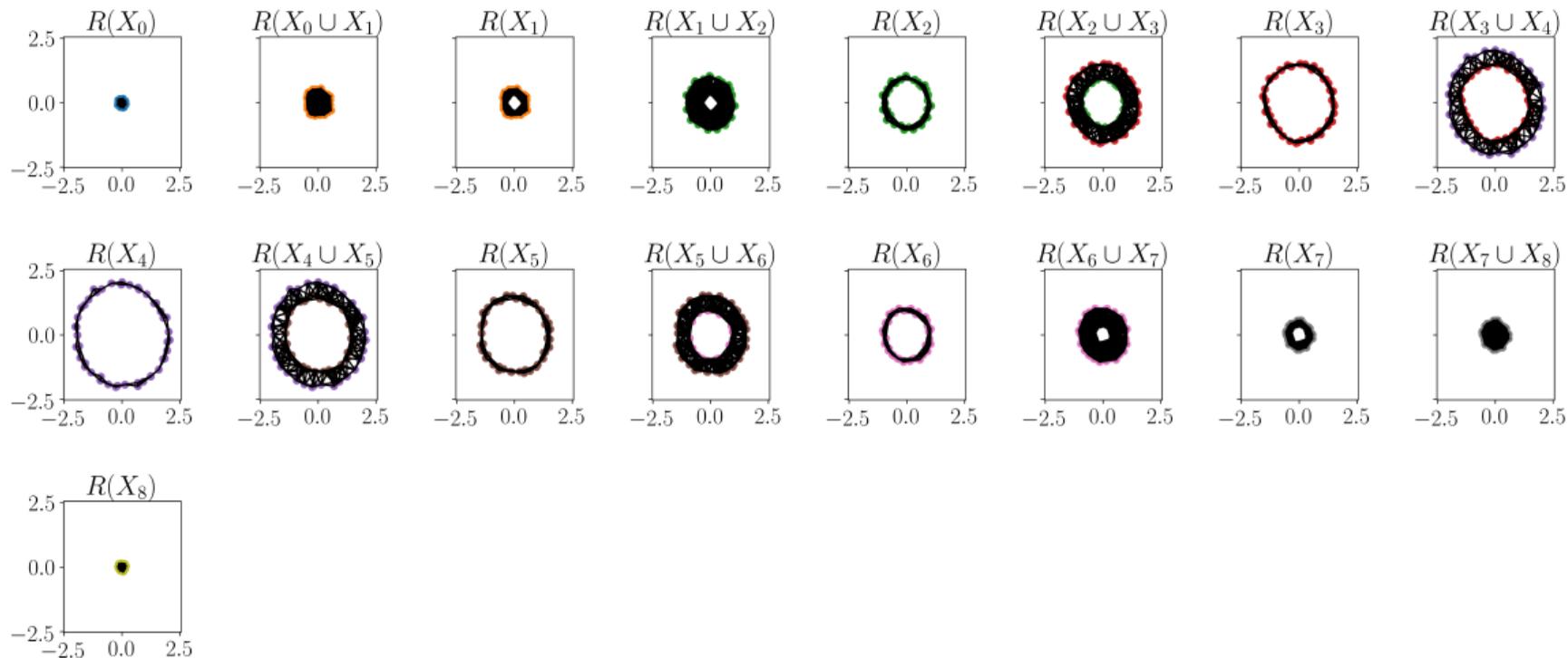
Time series to point clouds



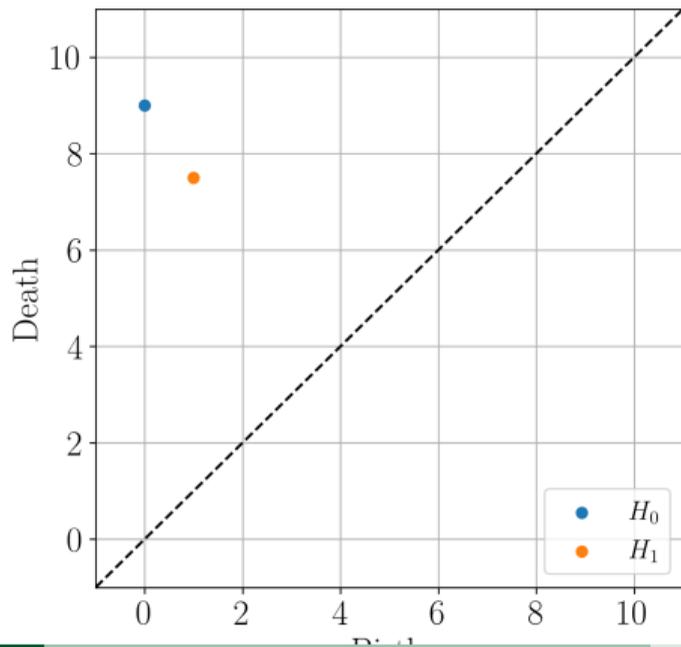
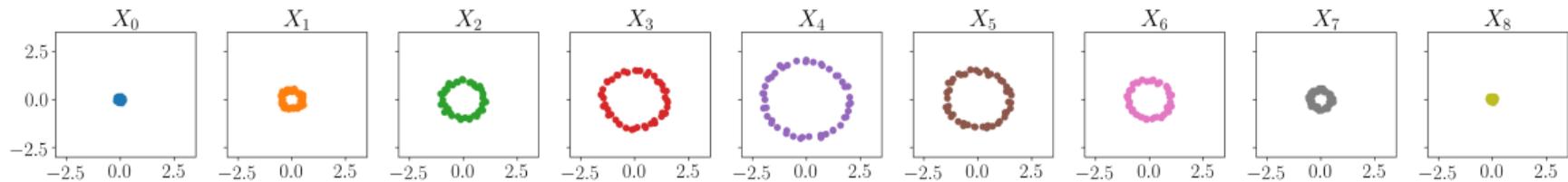
Time Delay
Embedding



Setting up the Zigzag



Result

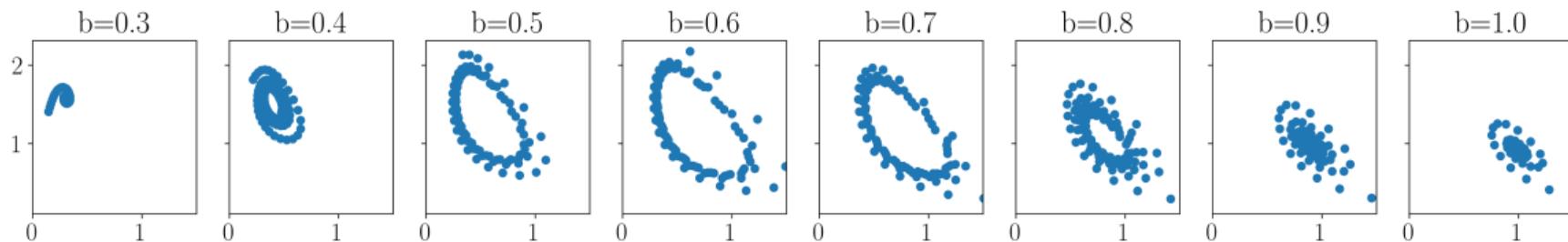


Can we detect changes in behavior of this dynamical system?

Sel'kov model for glycolysis:

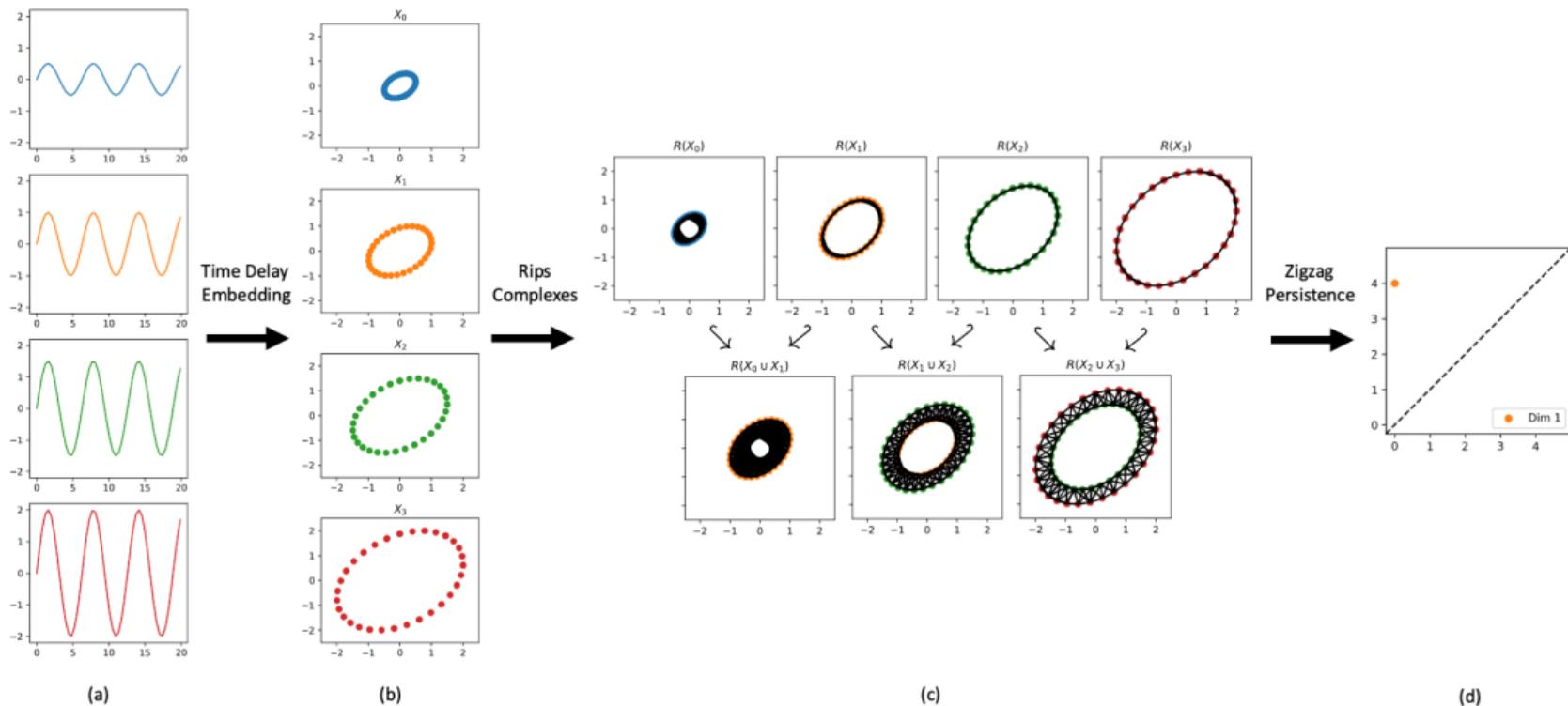
$$\frac{dx}{dt} = -x + ay + x^2y \qquad \frac{dy}{dt} = b - ay - x^2y$$

Specifically, can we detect for which values of b there is a Hopf bifurcation?

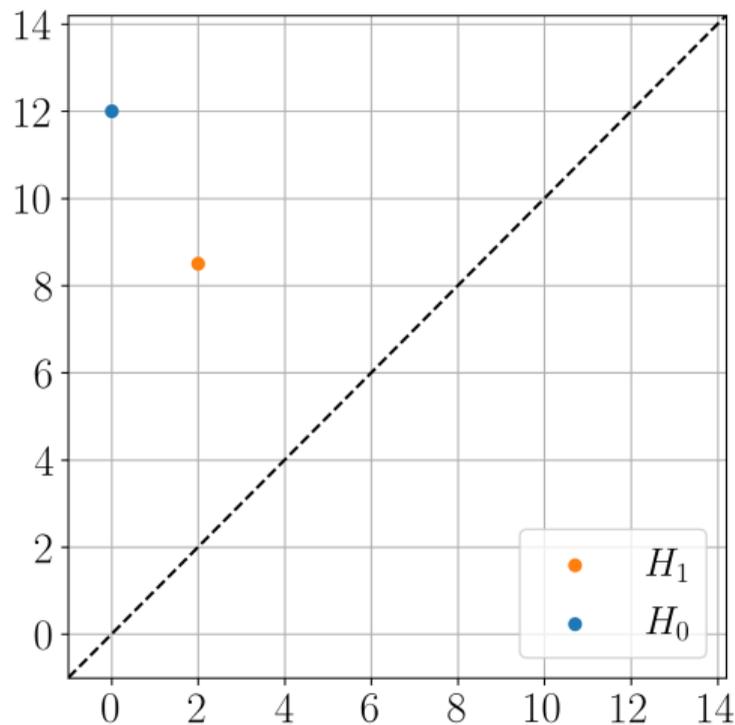
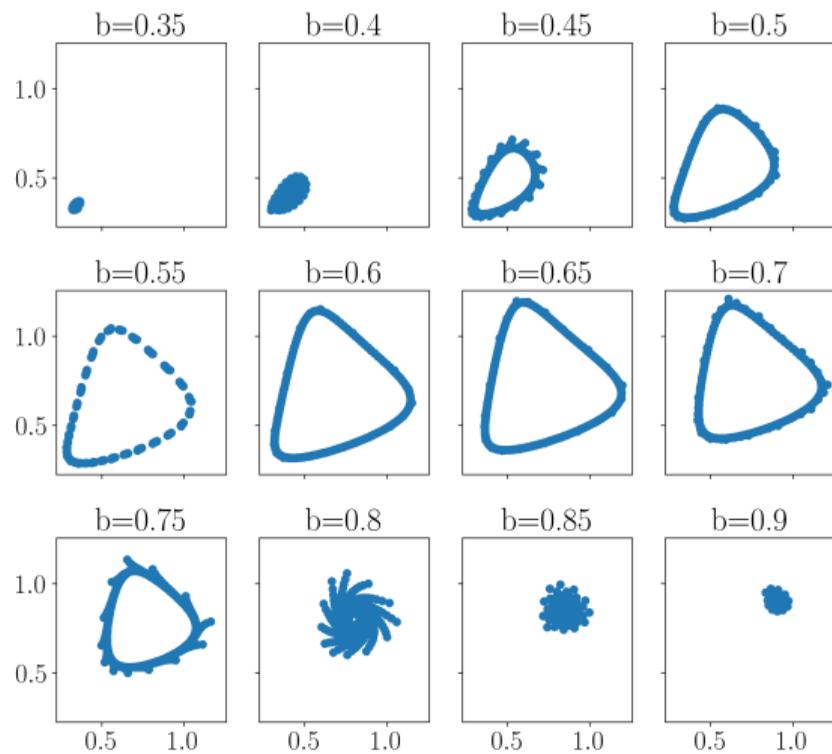


E. Sel'Kov. Self-oscillations in glycolysis. *European Journal of Biochemistry*, 4(1):79-86, 1968.

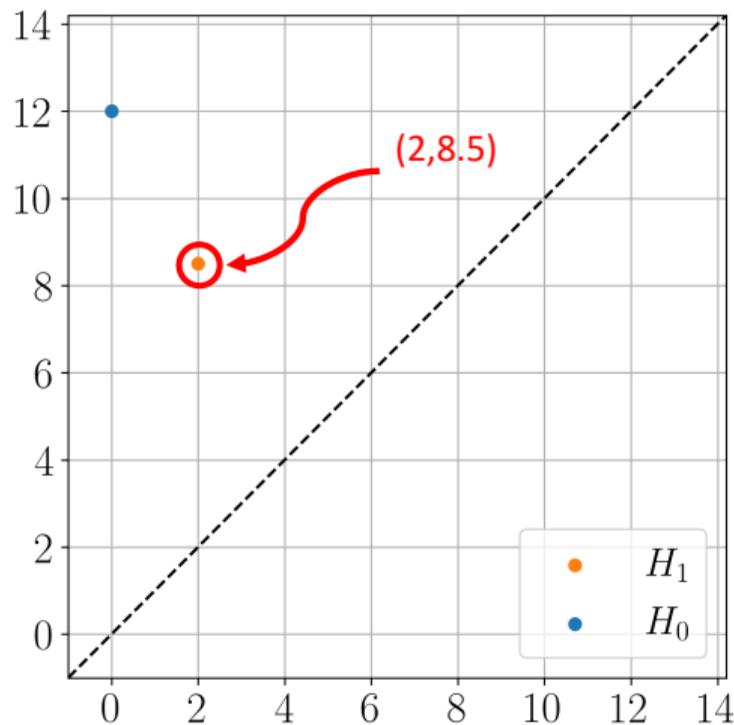
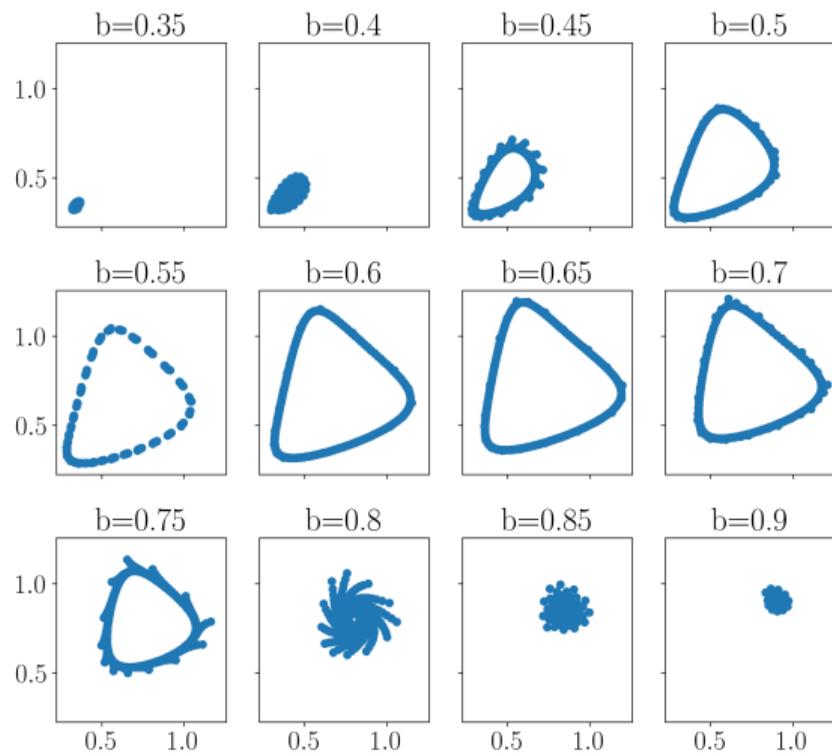
Overview of the Bifurcations using ZigZag (BuZZ) Method



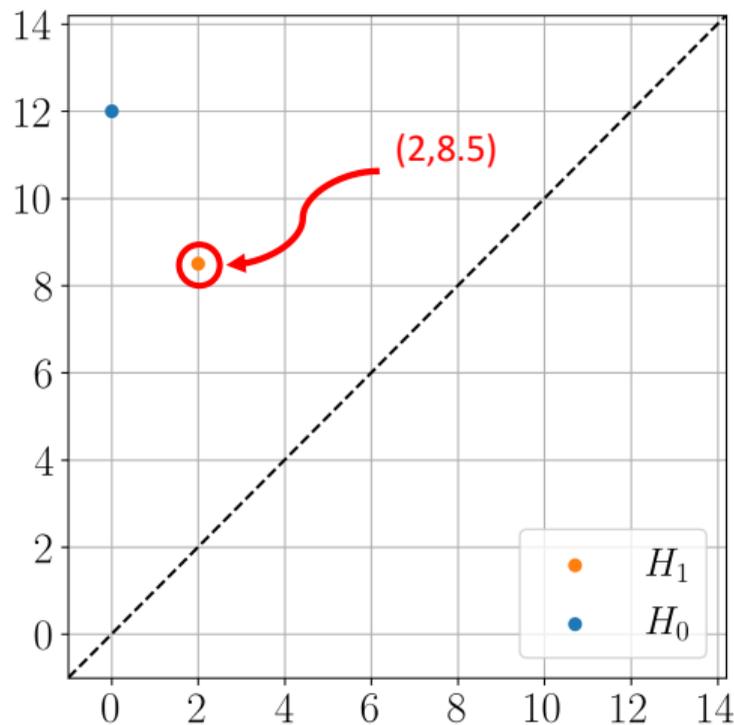
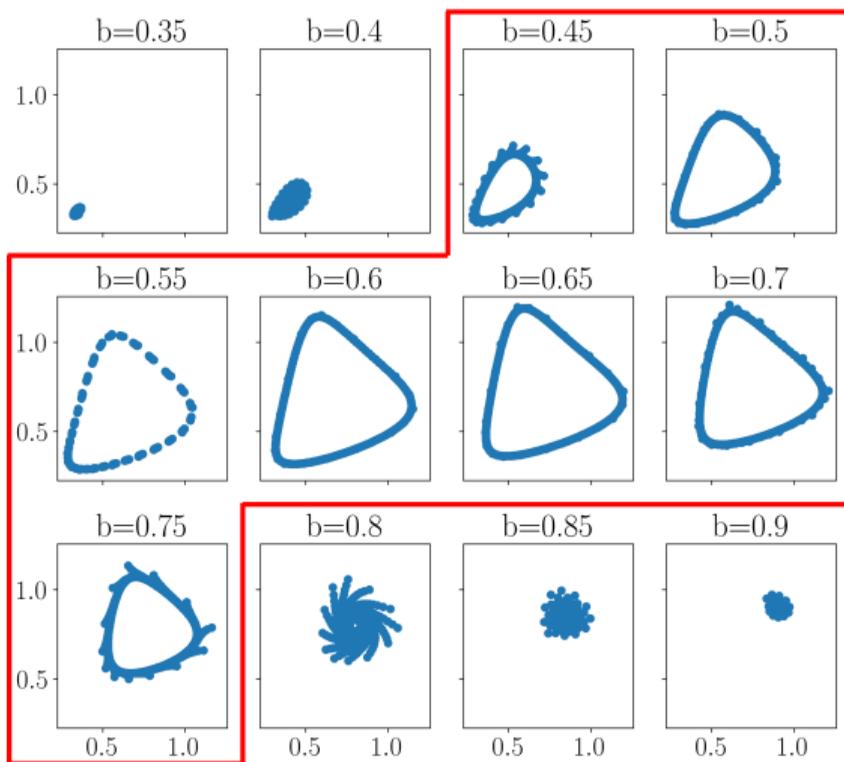
The result



The result



The result



We're done!

